3

SYSTEM FUNCTIONS

3.0 Introduction

The examples studied in the preceding chapter suggest that the frequency-domain description of the response of any lumped LTI circuit has a certain simple structure. In this chapter and the next we shall study this structure in some detail. Our study will not necessarily make it easier for us to determine the voltages and currents in any specific circuit problem. But our goals in this book go well beyond merely presenting efficient problem-solving techniques. Specifically, we hope to develop the insight and understanding necessary for the design of complex systems. For this, a full appreciation of the general properties of circuit behavior is even more important than skill at detailed circuit analysis.

3.1 A Superposition Formula for LTI Circuits

Example 2.5–2 led to a formula of the form

$$V_1(s) = H_1(s)I_0(s) + H_2(s)V_0(s) + H_3(s)\frac{v_C(0)}{s} + H_4(s)\frac{i_L(0)}{s} \tag{3.1–1}$$

for the transform $V_1(s)$ of the voltage across a particular pair of terminals in terms of the transforms $I_0(s)$ and $V_0(s)$ of two external sources and the transforms $v_C(0)/s$ and $i_L(0)/s$ of sources replacing the initial capacitor voltage and inductor current. The four functions $H_1(s)$, $H_2(s)$, $H_3(s)$, and $H_4(s)$, which relate the sources to $V_1(s)$, were derived by impedance methods.

It should be immediately evident from the Superposition Theorem of linear resistive circuit theory that the form of this result is general. That is, for any lumped LTI circuit we can always write a generalized *superposition formula*

$$Y(s) = \sum_{m=1}^{M} H_{em}(s)X_m(s) + \sum_{n=1}^{N} H_{in}(s)\frac{\lambda_n(0)}{s} \tag{3.1–2}$$

where

$Y(s)$ = $\mathcal{L}$-transform of the circuit voltage or current that is the object of our analysis, the one that we choose to designate as the *output* or *response* of the circuit;

$X_m(s)$ = $\mathcal{L}$-transform of the m^{th} independent external voltage or current source considered as an *input* or *stimulus* to the circuit; M is the number of external sources;

$\frac{\lambda_n(0)}{s}$ = $\mathcal{L}$-transform of the source describing the effect of the value $\lambda_n(0)$ of the n^{th} state variable at $t = 0$ (typically, a capacitor voltage or inductor current); N is the *order* of the system;

$H_{em}(s)$, $H_{in}(s)$ = functions of s relating each external source or initial condition source (respectively) to the output.

Equation (3.1–2) is the general form of the functional or operator description of LTI systems in the frequency domain that was promised in Section 2.1.

The two summation terms on the right in (3.1–2) are given separate names:

$$\sum_{n=1}^{N} H_{in}(s)\frac{\lambda_n(0)}{s} = \text{Zero Input Response (ZIR)} \tag{3.1–3}$$

$$\sum_{m=1}^{M} H_{em}(s)X_m(s) = \text{Zero State Response (ZSR).} \tag{3.1–4}$$

The ZIR term (loosely, the "free" or "natural" response) is not a function of inputs in the interval $t \geq 0$; it is determined by the initial state at $t = 0$, which in turn depends on inputs for $t < 0$. If all the external input sources are zero for $t \geq 0$, that is, if $X_m(s) = 0$, all m, then the ZIR is the total response for $t \geq 0$. On the other hand, the ZSR term (loosely, the "forced" or "driven" response) is not a function of the initial state. In particular, if the initial state is the *zero state*,* that is, if $\lambda_n(0) = 0$, all n, then the ZSR is the total response for all $t \geq 0$. The generalized superposition formula thus states that the total output at any time $t_0 \geq 0$ is a sum of the ZIR—the continuing effects at t_0 of inputs for $t < 0$—and the ZSR—the effects at t_0 of the inputs in the interval $0 \leq t < t_0$.

Note that we shall use the words "input" and "output" to refer either to the time functions, $x(t)$ and $y(t)$, or to their transforms, $X(s)$ and $Y(s)$; by the Uniqueness Theorem, these are alternative ways of describing the same things—the input or drive and the output or response. Note also that the ZIR and ZSR terms are not at all the same things as the "transient response" and the "steady-state response." Even if the input is a constant or a continuing sinusoid, so that "transient" and "steady-state" have unambiguous meanings, the ZSR term in general contains "transient" as well as "steady-state" components. We shall explore the relationships among these various response components more carefully in Section 3.3.

*It is perhaps worth pointing out that the zero state is a unique state for LTI systems in that it is independent of the choice of state variables (which is not unique).

3.2 System Functions

The separate elements making up the summations on the right in (3.1−2) all have the same structure—a product of the transform of a source $\big(X_m(s)$ or $\lambda_n(0)/s\big)$ and a function of s derived from the network $\big(H_{em}(s)$ or $H_{in}(s)\big)$. Thus we can interpret the factor $H_{em}(s)$ or $H_{in}(s)$ in each elementary term as the ratio of the $\mathcal{L}$-transform of the response component to the $\mathcal{L}$-transform of the source producing that component. Such a ratio of the transform of a response to the transform of a source is called a *system function*. System functions in electrical systems are classified as *driving-point* or *transfer* functions, accordingly as they relate voltages and currents at, respectively, the same or different *ports*.* They may be dimensionless *ratios* (voltage/voltage or current/current) or may have the dimensions of *impedance* (voltage/current) or *admittance* (current/voltage). Of course, since system functions may be used to relate drives and responses that are not voltages and currents—and not even electrical quantities—the range of possible dimensions is limitless.

Note, however, that system functions are always defined as output divided by input. To see why it is important to emphasize this point, consider the following example.

Example 3.2–1

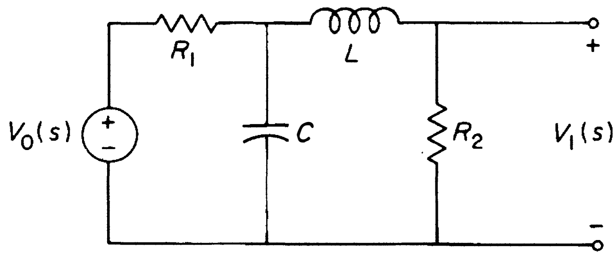

Figure 3.2–1. Circuit for Example 3.2–1.

The circuit shown in Figure 3.2–1 is part of the situation considered in Example 2.5–2 where we concluded that the ZSR term relating the output $V_1(s)$ and the input $V_0(s)$ is

$$V_1(s) = \frac{\dfrac{R_2}{R_1LC}}{s^2 + s\left(\dfrac{1}{R_1C} + \dfrac{R_2}{L}\right) + \dfrac{1}{LC}\left(1 + \dfrac{R_2}{R_1}\right)} V_0(s). \tag{3.2–1}$$

The system function is then given by

$$H_1(s) = \frac{\text{output}}{\text{input}} = \frac{V_1(s)}{V_0(s)} = \frac{\dfrac{R_2}{R_1LC}}{s^2 + s\left(\dfrac{1}{R_1C} + \dfrac{R_2}{L}\right) + \dfrac{1}{LC}\left(1 + \dfrac{R_2}{R_1}\right)}. \tag{3.2–2}$$

*For a discussion of the concept of a *port* (loosely, a pair of terminals) see the appendix to this chapter.

$H_1(s)$ is a (dimensionless) transfer function since $V_1(s)$ and $V_0(s)$ are defined at different ports.

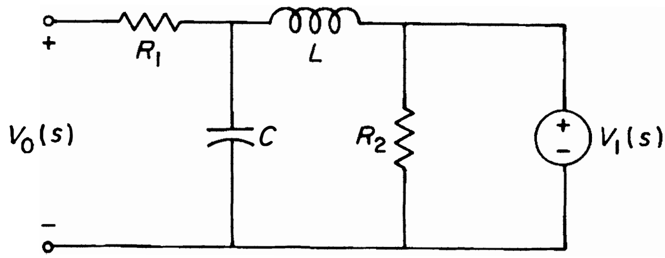

Figure 3.2–2. Circuit of Figure 3.2–1 with source at right.

Suppose, however, we were to drive this circuit with a voltage source at the right-hand port and measure the voltage response at the left as shown in Figure 3.2–2. We readily compute that in this case (which was not considered in Example 2.5–2) the ZSR term in $V_0(s)$ (which now represents the output) determined by $V_1(s)$ (which now represents the input) is

$$V_0(s) = \frac{\frac{1}{Cs}}{\frac{1}{Cs} + Ls} V_1(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{LC}} V_1(s) \qquad (3.2\text{–}3)$$

which corresponds to the system function

$$H_2(s) = \frac{\text{output}}{\text{input}} = \frac{V_0(s)}{V_1(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{LC}}. \qquad (3.2\text{–}4)$$

Note that $H_2(s)$ is a ratio of the voltage across the left-hand pair of terminals to the voltage across the right-hand pair; so is $1/H_1(s)$ as defined above. But $1/H_1(s)$ and $H_2(s)$ are totally different!

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The point of this example is to show that it is generally necessary to identify not only what pair of variables in a circuit are related by a system function, but also which variable is the source and which the response. The simplest way to make this evident is to adopt the convention that system functions are always defined as output divided by input. Consequently, the reciprocal of a system function is not necessarily a system function. There is, however, one very important case in which a system function correctly describes the relationship between two variables irrespective of which is the drive and which the response, as illustrated in the next example.

Example 3.2–2

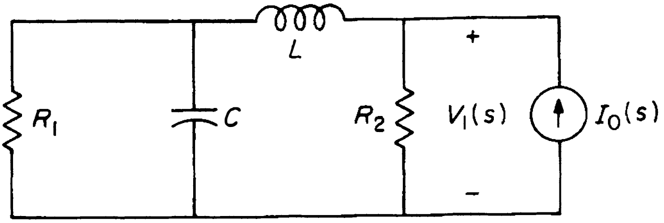

Figure 3.2–3. Circuit for Example 3.2–2.

The circuit shown in Figure 3.2–3 is also part of the situation considered in Example 2.5–2, where we concluded that the ZSR term relating the output $V_1(s)$ and the input $I_0(s)$ is

$$V_1(s) = \frac{R_2\left(s^2 + \dfrac{s}{R_1C} + \dfrac{1}{LC}\right)}{s^2 + s\left(\dfrac{1}{R_1C} + \dfrac{R_2}{L}\right) + \dfrac{1}{LC}\left(1 + \dfrac{R_2}{R_1}\right)} I_0(s). \tag{3.2–5}$$

The system function is then given by

$$H_3(s) = \frac{\text{output}}{\text{input}} = \frac{V_1(s)}{I_0(s)} = \frac{R_2\left(s^2 + \dfrac{s}{R_1C} + \dfrac{1}{LC}\right)}{s^2 + s\left(\dfrac{1}{R_1C} + \dfrac{R_2}{L}\right) + \dfrac{1}{LC}\left(1 + \dfrac{R_2}{R_1}\right)}. \tag{3.2–6}$$

$H_3(s)$ is a driving-point impedance since $V_1(s)$ and $I_0(s)$ are the voltage and current at the same port.

Suppose, however, we were to drive this circuit with a voltage source instead of a current source and measure the current response instead of the voltage, as shown in Figure 3.2–4.

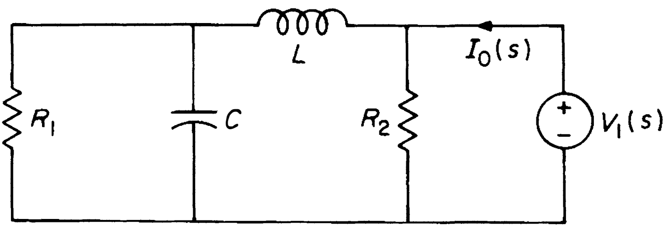

Figure 3.2–4. Circuit of Figure 3.2–3 driven by a voltage source.

We readily compute

$$I_0(s) = \frac{V_1(s)}{R_2} + \frac{V_1(s)}{Ls + \dfrac{R_1/Cs}{R_1 + 1/Cs}}$$

$$= \frac{s^2 + s\left(\dfrac{1}{R_1C} + \dfrac{R_2}{L}\right) + \dfrac{1}{LC}\left(1 + \dfrac{R_2}{R_1}\right)}{R_2\left(s^2 + \dfrac{s}{R_1C} + \dfrac{1}{LC}\right)} V_1(s). \qquad (3.2\text{–}7)$$

The system function is then given by

$$H_4(s) = \frac{\text{output}}{\text{input}} = \frac{I_0(s)}{V_1(s)} = \frac{s^2 + s\left(\dfrac{1}{R_1C} + \dfrac{R_2}{L}\right) + \dfrac{1}{LC}\left(1 + \dfrac{R_2}{R_1}\right)}{R_2\left(s^2 + \dfrac{s}{R_1C} + \dfrac{1}{LC}\right)} \qquad (3.2\text{–}8)$$

and is a driving-point admittance.

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Note that in this example, unlike the situation with transfer system functions, interchanging which variable is considered the input and which the output simply inverts the system function. This is evidently a general property of driving-point impedances and admittances. Can you explain why driving-point and transfer system functions are different in this respect? The driving-point impedance or admittance characterizes the behavior of a 2-terminal (single-port) LTI network no matter how it is driven or connected externally. Similar comprehensive descriptions are possible for multiterminal LTI networks designed for their transfer properties, but such descriptions require more than the specification of a single transfer function (see the appendix to this chapter).

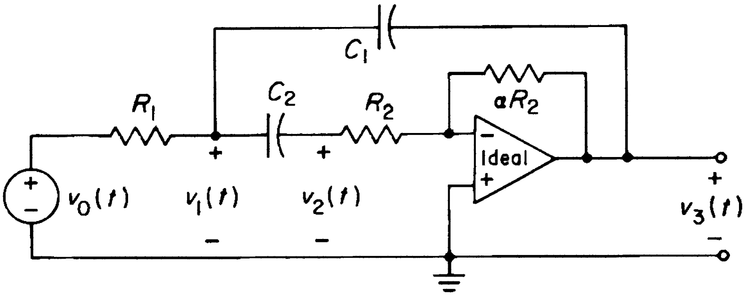
3.3 System Functions as Response Amplitudes to Exponential Drives

As explained in Chapter 2, system functions are readily computed by impedance methods with L and C replaced by impedances Ls and $1/Cs$, and the network then solved as if it were a resistive circuit. This same procedure—with s replaced by $j\omega$—is used (as you know from earlier studies) to find the *sinusoidal steady-state frequency response* of a network. Thus, for $s = j\omega$, the system function $H(s)$ has the following interpretation:

> If the input to an LTI network is the complex exponential $Xe^{j\omega t}$ and the steady-state output is the complex exponential $Ye^{j\omega t}$, then
>
> $$Y/X = H(j\omega)$$
>
> where $H(s)$ is the system function relating the input and output.

In other words, $e^{j\omega t}$ as a drive gives $H(j\omega)e^{j\omega t}$ as the "steady-state" response after the "transients" have died away. Indeed, this result is commonly used as the basis for the experimental measurement of $H(j\omega)$ for an LTI system whose internal structure is unknown, concealed inside a "black box." We now show with an example that $H(s)$ has a similar interpretation even when $s \neq j\omega$, so that $e^{st} = e^{\sigma t}e^{j\omega t}$ cannot correspond to any straightforward notion of "steady-state" (since $e^{\sigma t}$, $\sigma \neq 0$, either grows or decays with time).

Example 3.3–1



$$R_1 = R_2 = 1\,\Omega, \quad C_1 = \frac{2}{3}\text{ F}, \quad C_2 = \frac{1}{3}\text{ F}, \quad \alpha = \frac{1}{2}$$

Figure 3.3–1. Circuit for Example 3.3–1.

In Problem 3.1, you will show that $H(s)$ for the circuit in Figure 3.3–1 is

$$H(s) = \frac{V_3(s)}{V_0(s)} = \frac{\dfrac{-\alpha}{(1+\alpha)R_1C_1}s}{s^2 + \dfrac{s}{1+\alpha}\left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_2} + \dfrac{1}{R_2C_1}\right) + \dfrac{1}{(1+\alpha)R_1C_1R_2C_2}} \tag{3.3–1}$$

which for the given values becomes

$$H(s) = \frac{-s/2}{s^2+4s+3} = \frac{-s/2}{(s+1)(s+3)}. \tag{3.3–2}$$

Now let $v_0(t) = e^{s_0 t}$, $t > 0$, where s_0 is an arbitrary complex number. Then $\mathcal{L}[v_0(t)] = V_0(s) = 1/(s-s_0)$, and under ZSR conditions

$$V_3(s) = H(s)V_0(s) = \frac{-s/2}{(s+1)(s+3)(s-s_0)}. \tag{3.3–3}$$

If $s_0 \neq -1, -3$, we may expand

$$V_3(s) = \frac{\dfrac{-1}{4(1+s_0)}}{s+1} + \frac{\dfrac{3}{4(3+s_0)}}{s+3} + \frac{\dfrac{-s_0/2}{(s_0+1)(s_0+3)}}{s-s_0} \tag{3.3–4}$$

where we recognize the residue in the $(s - s_0)$ term as $H(s_0)$ so that the ZSR is

$$v_3(t) = -\frac{1}{4(1+s_0)}e^{-t} + \frac{3}{4(3+s_0)}e^{-3t} + H(s_0)e^{s_0 t}, \quad t > 0. \tag{3.3–5}$$

If s_0 is purely imaginary, $s_0 = j\omega_0$, this result describes the usual sinusoidal-drive situation—the first two terms are "transients" resulting from the sudden application to a resting network of the drive $e^{j\omega_0 t}$ at $t = 0$; in time they die away, leaving the "steady-state" response $H(j\omega_0)e^{j\omega_0 t}$. But even if $s_0 \neq j\omega_0$, the last term—the "driven" term—in the expression for $v_3(t)$ will come to dominate after a while provided that $\Re e[s_0] > -1$. (If $\Re e[s_0] < -1$, the "driven" term vanishes more rapidly than the "transient" terms; the latter thus ultimately become relatively more important, although all three terms may be decaying. Note also that the word "driven" here in quotes means simply that term having the same form as the drive—that is, $H(s_0)e^{s_0 t}$. The total driven (zero-state) response contains both the "driven" term and "transient" terms.)

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The result obtained in Example 3.3–1 is general:

> If the input to an LTI system has the form $e^{s_0 t}$, then the output will become predominantly $H(s_0)e^{s_0 t}$ as time passes, provided that s_0 lies in that part of the s-plane to the right of the rightmost pole of $H(s)$. This region is called the *domain of convergence* for $H(s)$.

This important observation illustrates the complete way in which $H(s)$ is a generalization of $H(j\omega)$, and justifies calling s the *complex frequency*.

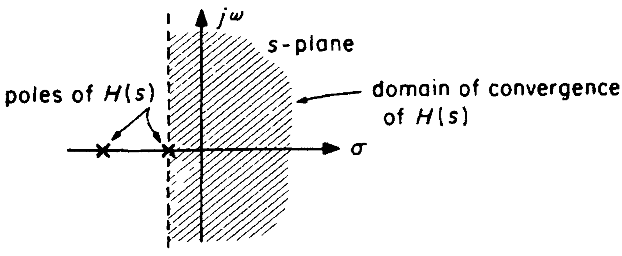

Figure 3.3–2. Domain of convergence.

3.4 System Functions and the Input-Output Differential Equation

There is a complete and close relationship between the system function $H(s)$ and the input-output differential equation obtained from the node or state differential equations by eliminating all of the unknown variables except the output. Several examples will make the general relationship evident.

Example 3.4–1

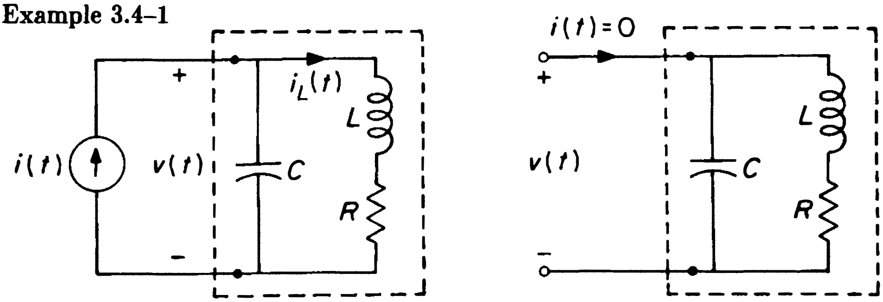

Figure 3.4–1. Circuit for Example 3.4–1.

Differential state equations for the circuit to the left in Figure 3.4–1 in terms of $v(t)$ and $i_L(t)$, the capacitor voltage and inductor current, are

$$L\frac{di_L(t)}{dt} = -Ri_L(t) + v(t) \tag{3.4–1}$$

$$C\frac{dv(t)}{dt} = i(t) - i_L(t). \tag{3.4–2}$$

Solving (3.4–2) for $i_L(t)$ and substituting into (3.4–1) give the input-output equation in terms of the drive $i(t)$ and the response $v(t)$:

$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{1}{C}\left(\frac{di(t)}{dt} + \frac{R}{L}i(t)\right). \tag{3.4–3}$$

The natural frequencies of this circuit—that is, the frequencies present in $v(t)$ when $i(t) = 0$ and the terminals are open-circuit as shown to the right in Figure 3.4–1—are the roots of the characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{3.4–4}$$

derived from the left-hand side of the input-output equation.

The system function relating the drive and the response is the driving-point impedance

$$Z(s) = \frac{V(s)}{I(s)} = \frac{\text{output}}{\text{input}} \tag{3.4–5}$$

which is readily found by series and parallel impedance arguments to be

$$Z(s) = \frac{\frac{1}{Cs}(Ls + R)}{\frac{1}{Cs} + Ls + R} = \frac{\frac{1}{C}\left(s + \frac{R}{L}\right)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{V(s)}{I(s)}. \tag{3.4–6}$$

Cross-multiplying gives

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)V(s) = \frac{1}{C}\left(s + \frac{R}{L}\right)I(s). \tag{3.4–7}$$

Comparison with the input-output differential equation shows that here, and in general, to derive the input-output differential equation from the system function it is only necessary to cross-multiply and identify $d/dt \leftrightarrow s$. Often, indeed, the easiest way to derive the input-output differential equation is to use impedance methods first to find the system function. The tricky process of eliminating intermediate veriables and their derivatives is thus much simplified. Algebraic equations are easier to manipulate than differential equations.

The relationship between the system function and the input-output differential equation can also be deduced in the opposite direction by exploiting the result derived in Section 3.3—that e^{st} as an input gives ultimately the output $H(s)e^{st}$ for s in the domain of convergence. Thus if we substitute $i(t) = e^{st}$ and $v(t) = H(s)e^{st}$ into the input-output differential equation above, we obtain

$$H(s)s^2e^{st} + H(s)\frac{R}{L}se^{st} + H(s)\frac{1}{LC}e^{st} = \frac{1}{C}\left(se^{st} + \frac{R}{L}e^{st}\right). \qquad (3.4\text{–}8)$$

Solving for $H(s)$ gives the same result $H(s) = Z(s)$ as derived above by impedance methods.*

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An important conclusion to be derived from the relationship between the system function and the input-output differential equation is that the natural frequencies of the circuit are the roots of the denominator polynomial, that is, the poles of $H(s)$.† Loosely, the output $Y(s) = H(s)X(s)$ can be finite when the input $X(s)$ is zero only if $H(s)$ is infinite, and this happens only for the values of s that are the poles of $H(s)$.

Example 3.4–2

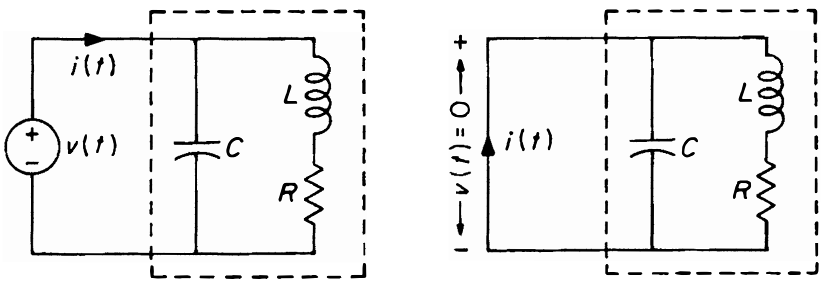

Figure 3.4–2. Circuit for Example 3.4–2.

*Attempts to derive the form of the ZSR and $H(s)$ from the input-output differential equation by direct application of the $\mathcal{L}$-transform Differentiation Theorem can lead to difficulties. See Problem 3.3.

†If the numerator and denominator polynomials of $H(s)$ contain a common factor, that is, if a zero of $H(s)$ cancels a pole, the relationship between $H(s)$ and the ZIR is less close. Specifically, the ZIR may contain a term whose complex frequency is not a pole of $H(s)$. However, there is no signal $x(t)$, $t < 0$, that when applied to the normal input corresponding to $H(s)$, will generate such a ZIR term for $t > 0$. On the other hand, if such a ZIR term is excited, for example, by driving the network at some other input, then there is no input that can be applied at the normal input that will cancel the effects of this ZIR term in a finite time; such a system is said to be *uncontrollable*. For an example, see Exercise 4.5.

Note that the poles of $H(s)$ are the natural frequencies under the condition that the input is zero. If the input is a current source as in Example 3.4–1, then zero input implies that the input terminals are open-circuit. But if the input is a voltage source, zero input implies that the input terminals are shorted. Thus consider (as shown in Figure 3.4–2) the same circuit as before but driven by a voltage source $v(t)$; the response will now be taken to be the current $i(t)$. The input-output roles of $v(t)$ and $i(t)$ are interchanged. Hence the characteristic equation is

$$\frac{1}{C}\left(s+\frac{R}{L}\right)=0. \qquad (3.4\text{–}9)$$

The root of the characteristic equation, $s=-R/L$, defines the functional form of the current, $i(t)=Ie^{-Rt/L}$, that can flow under short-circuit conditions, that is, $v(t)=0$, as shown to the right in Figure 3.4–2. The system function in this case is the driving-point *admittance**

$$Y(s)=\frac{I(s)}{V(s)}. \qquad (3.4\text{–}10)$$

Of course,

$$Y(s)=\frac{1}{Z(s)}=\frac{C\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)}{s+\frac{R}{L}}. \qquad (3.4\text{–}11)$$

The natural frequency is the pole of $Y(s)$, or the zero of $Z(s)$, under short-circuit conditions.

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The observation that the poles of the system function $H(s)$ are the natural frequencies of the circuit is, of course, consistent with the fact that

$$Y(s)=H(s)X(s) \qquad (3.4\text{–}12)$$

is the $\mathcal{L}$-transform of the ZSR output $y(t)$ to the input $x(t)$. If $Y(s)=\mathcal{L}[y(t)]$ is expanded in partial fractions, the poles of $H(s)$ yield terms describing that part of the response whose form is determined by the circuit rather than the drive.

3.5 Summary

The response of a linear time-invariant circuit can always be interpreted as the sum of the zero-state response and the zero-input response. Each of these components is in turn a superposition of terms describing the separate effects of each of the external sources and each of the internal sources reflecting the initial state. Each term has, in the frequency domain, the form of a product of the $\mathcal{L}$-transform of the n^{th} source times a function of s, $H_n(s)$, called the system

*The word "admittance" was coined by Heaviside to describe the ratio of a current to a voltage. Regrettably, the symbol $Y(s)$ for admittance is well-established. Note the distinction in this book between $Y(s)$ as the admittance in the electric-circuit example leading to (3.4–11) and $Y(s)$ as the $\mathcal{L}$-transform of $y(t)$ in a general formula such as (3.4–12).

function, which relates the response to that particular source or drive. System functions are usually easy to determine by impedance methods from a structural description of the circuit. If the particular drive has the form $e^{s_0 t}$, then the response component will approach $H_n(s_0)e^{s_0 t}$ after a while if s_0 is in the domain of convergence for $H_n(s)$, that is, in the region to the right of the rightmost pole of $H_n(s)$. Since the poles of $H_n(s)$ are the natural frequencies of the system, $e^{s_0 t}$ under these conditions decays more slowly (or grows more rapidly) than the natural response terms, and this explains its eventual dominance. Finally, $H_n(s)$ and the input-output differential equation relating the response to that particular source contain essentially identical information. Thus, in general, the system function summarizes everything there is to know about the input-output behavior of an LTI system.

APPENDIX TO CHAPTER 3

System Function Characterization of LTI 2-Ports

As illustrated in Example 3.2–1, a system function describes the behavior of a network only under certain specified source and termination conditions. Complete "black-box" descriptions of multiterminal LTI networks adequate to characterize the behavior of the network under any conditions of drive and load are possible, however, and are often extremely useful. An important special case is discussed in this appendix.

Consider a network composed of linear time-invariant R's, L's, and C's, ideal transformers, controlled sources, etc., enclosed in a box so that it is accessible only through four terminals. Suppose further that the external connections to the 4-terminal network constrain the terminal currents to be *paired*—the currents in each pair being equal in magnitude and opposite in sign. Such a 4-terminal network is called a *2-port.** One external arrangement that ensures 2-port behavior is shown in Figure 3.A–1. Obviously many other possibilities exist; a sufficient condition is that there be no connection between the external networks connected to ports 1 and 2 except through the 2-port. The description of a 4-terminal network as a 2-port obviously does not characterize the behavior of the network under all external conditions, but it is adequate for many purposes.

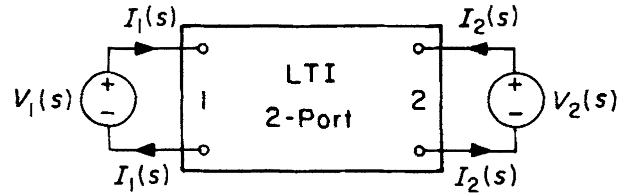


Figure 3.A–1.

Even if we do not know what the detailed circuit arrangements may be inside an LTI 2-port (that is, if we are forced to consider the 2-port as a "black box" accessible only through electrical measurements made at its terminal pairs), we can still conclude

*Extending this terminology, a network connected to the rest of the world through n paired terminals is called an *n-port.* A 2-terminal network is always a 1-port, since no matter how it is connected, KCL guarantees that the currents at the two terminals are paired. A 3-terminal network may always be represented as a 2-port without loss of generality, as shown in Figure 3.A–2.

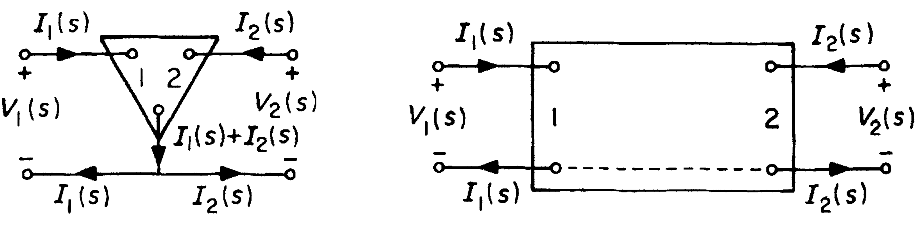
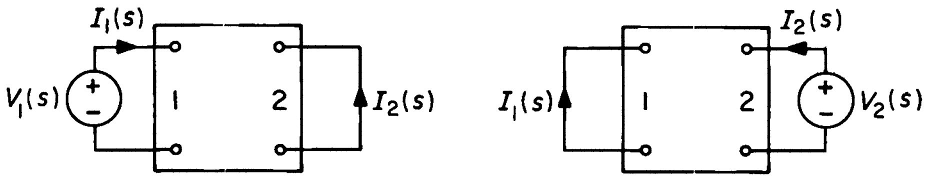


Figure 3.A–2.

from superposition and impedance arguments such as discussed in this chapter that the ZSR currents $I_1(s)$ and $I_2(s)$ must be given by equations of the form

$$I_1(s) = Y_{11}(s)V_1(s) + Y_{12}(s)V_2(s)$$
$$I_2(s) = Y_{21}(s)V_1(s) + Y_{22}(s)V_2(s)$$

in terms of the external voltage sources $V_1(s)$ and $V_2(s)$. The various system functions $Y_{ij}(s)$ are called the *short-circuit driving-point* and *transfer admittances* of the 2-port because they can, at least in principle, be inferred from measurements made on the circuit with one or the other of the terminal pairs shorted, as shown in Figure 3.A–3.



$$Y_{11}(s) = \left.\frac{I_1(s)}{V_1(s)}\right|_{V_2(s)=0} \qquad Y_{12}(s) = \left.\frac{I_1(s)}{V_2(s)}\right|_{V_1(s)=0}$$

$$Y_{21}(s) = \left.\frac{I_2(s)}{V_1(s)}\right|_{V_2(s)=0} \qquad Y_{22}(s) = \left.\frac{I_2(s)}{V_2(s)}\right|_{V_1(s)=0}$$

Figure 3.A–3.

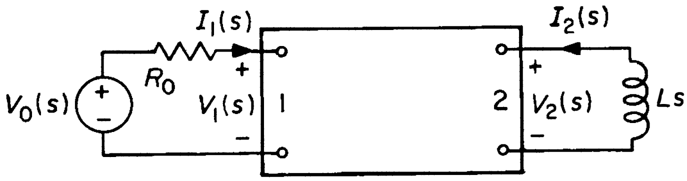


Figure 3.A–4.

The short-circuit admittances characterize the ZSR behavior of the 2-port under any external conditions that satisfy the paired-current condition. For example, if the 2-port is loaded with an inductor at port 2 and driven from a circuit with Thévenin parameters as shown in Figure 3.A–4, then the source and load impose the conditions

$$V_1(s) = V_0(s) - I_1(s)R_0$$
$$V_2(s) = -LsI_2(s).$$

Combining with the two short-circuit admittance equations, we may eliminate $V_1(s)$, $I_1(s)$, and $I_2(s)$ to obtain the overall transfer ratio under these conditions

$$\frac{V_2(s)}{V_0(s)} = \frac{-\frac{1}{R_0}Y_{21}(s)}{\left(\frac{1}{R_0} + Y_{11}(s)\right)\left(\frac{1}{Ls} + Y_{22}(s)\right) - Y_{12}(s)Y_{21}(s)}.$$

(See Problem 3.6 for another way of deriving this result. If the 2-port is reciprocal, the derivation is even simpler; see Problem 3.5. Many alternate methods of characterizing a 2-port are possible, one or another of which may be simpler in a particular situation. See Problems 3.7 and 3.8 for examples.)

If limitations are imposed on the kinds of elements out of which the 2-port is constructed, then the $Y_{ij}(s)$ will in general have to satisfy certain conditions—some of which will be discussed in Chapter 4. One of the more interesting conditions is *reciprocity*,

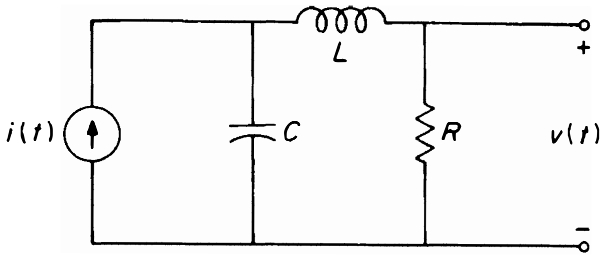
$$Y_{12}(s) = Y_{21}(s)$$

which is guaranteed if the 2-port contains LTI R's, L's, C's, and transformers, but no controlled sources. For a further discussion of reciprocity, see Problem 3.5.

EXERCISES FOR CHAPTER 3

Exercise 3.1

Consider the circuit below in which the current source is the input and the voltage across the resistor R is the output:



a) Show that the system function relating the input and output is

$$H(s) = \frac{\frac{R}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.$$

What are the dimensions of the separate terms in the denominator and numerator? What are the dimensions of $H(s)$?

b) Determine the input-output differential equation.

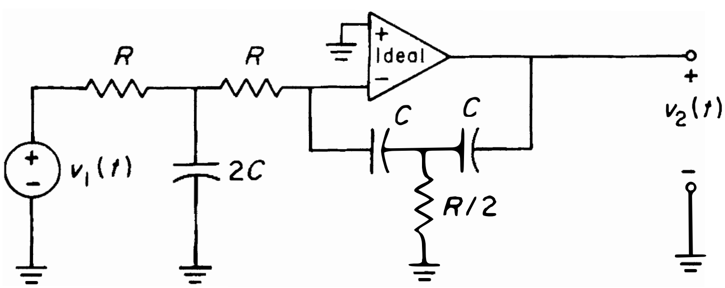
c) Show that the form of the ZIR is

$$v(t) = Ae^{-t/2}\cos\left(\frac{\sqrt{3}t}{2} + \theta\right)$$

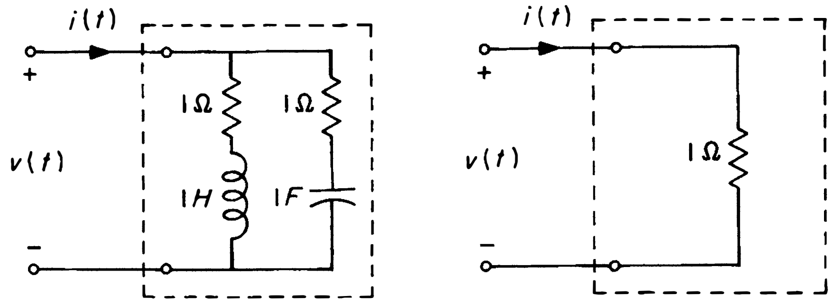
if $R = 1\,\Omega$, $L = 1$ H, $C = 1$ F.

Exercise 3.2

Use impedance methods to derive the input-output differential equation for the circuit below and thus show that it behaves as a *double integrator*, that is, $v_1(t) \sim d^2 v_2(t)/dt^2$.



Exercise 3.3



a) Show that the driving-point impedance of the network on the left is identical for all s to that on the right; that is,

$$Z(s) = \frac{V(s)}{I(s)} = 1.$$

b) Many years ago Joseph Slepian, writing in the *Transactions* of the old American Institute of Electrical Engineers, proposed as a puzzle the description of a test to be performed solely at the electrical terminals that would permit you to discover which of the two "black boxes" above you had been handed. A flood of letters resulted and the argument went on for months. Some writers tried to prove that no successful test was possible; others maintained that under certain excitation conditions the circuits would behave differently. What is your position?

Exercise 3.4

Experiments on an LTI system lead to the following conclusions:

a) Independent of the state of the system at $t = 0$, an input $x(t) = e^{-2t}$, $t > 0$, yields an output of the form $y(t) = 3e^{-2t} + (k_0 + k_1 t)e^{-t}$, $t > 0$;

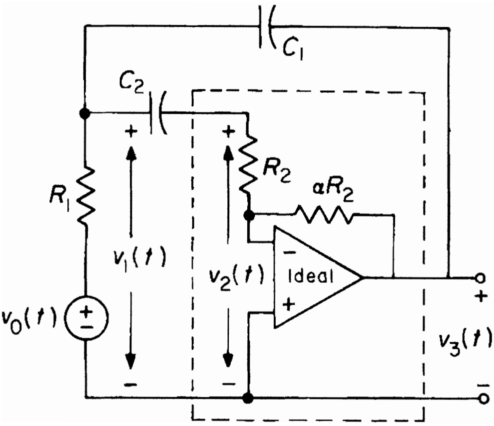
b) Independent of the state of the system at $t = 0$, an input $x(t) = e^{-3t}$, $t > 0$, yields an output of the form $y(t) = (k_2 + k_3 t)e^{-t}$, $t > 0$.

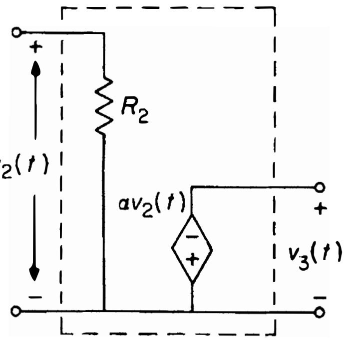
Argue that, if the system function $H(s)$ is a proper fraction ($H(s) \to 0$ as $s \to \infty$), it must be

$$H(s) = \frac{3(s+3)}{(s+1)^2}.$$

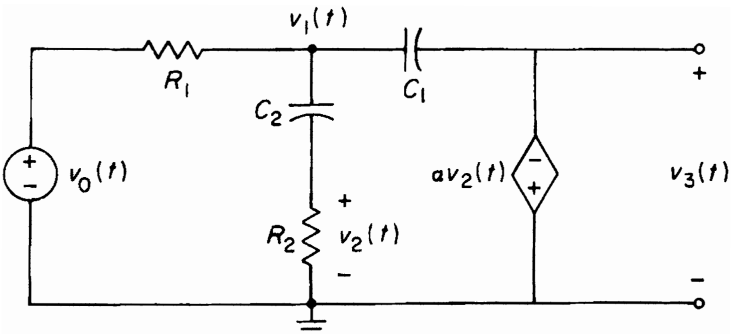
PROBLEMS FOR CHAPTER 3

Problem 3.1





a) Argue that the dashed section in the circuit above is equivalent to the circuit shown to the right, so that an overall equivalent circuit takes the form shown below.

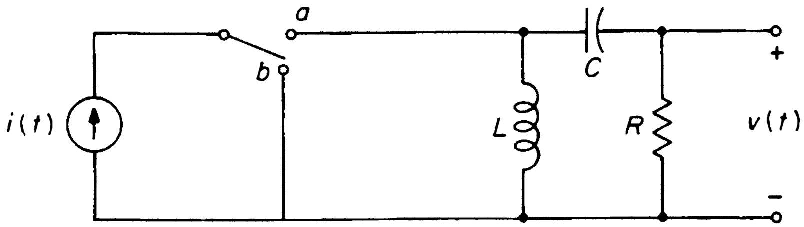


b) Using impedance methods, write ZSR node equations for the nodes whose voltages are labelled $v_1(t)$ and $v_2(t)$. Show that your results are consistent with the node equations in differential form given in Exercise 1.4.

c) Solve these node equations for the system function $H(s) = V_3(s)/V_0(s)$ and check your result with the formula given in Example 3.3–1.

d) Take the $\mathcal{L}$-transform of the differential state equations given in Exercise 1.4 under zero-state conditions. Solve these equations for $H(s) = V_3(s)/V_0(s)$, where $V_3(s) = -\alpha V_2(s)$, and compare your result with that derived in (c).

Problem 3.2

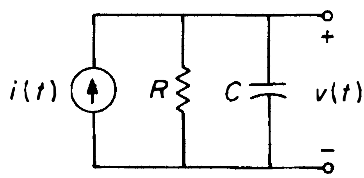


$C = 0.01\ \mu\text{F},\ R = 3\ \text{k}\Omega,\ L = 20\ \text{mH}$

a) With the switch in position a, find the system function $H(s) = V(s)/I(s)$, where $v(t)$ is the ZSR to the input $i(t)$, $V(s) = \mathcal{L}[v(t)]$, etc.
b) Suppose the current $i(t)$ has been held at a constant value $i(t) = 1$ mamp with the switch in position a for a very long time prior to $t = 0$. At $t = 0$ the switch is moved to position b. Find the values of $v(t)$ and $dv(t)/dt$ just after the switch is changed.
c) Find the natural frequencies of the circuit.
d) Find $v(t)$ for all time after the switch is switched to b.

Problem 3.3

Lynn Iyar, one of the more mathematically inclined students in the class, was not satisfied with the rather informal ways described in Section 3.4 for relating the input-output differential equation and the system function for an LTI circuit. Why not, she thought, simply apply the $\mathcal{L}$-transform Differentiation Theorem to obtain the desired relationship directly? Thus the RC circuit shown to the right corresponds to the input-output differential equation



$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) = i(t). \tag{1}$$

Applying the $\mathcal{L}$-transform Differentiation Theorem,

$$\mathcal{L}\left[\frac{dx(t)}{dt}\right] = s\mathcal{L}[x(t)] - x(0) \tag{2}$$

yields

$$CsV(s) - v(0) + \frac{1}{R}V(s) = I(s). \tag{3}$$

Since the system function describes the ZSR, Lynn argued we should set $v(0) = 0$ and solve for

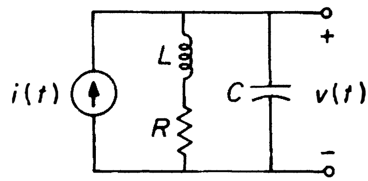
$$H(s) = \frac{V(s)}{I(s)} = \frac{\frac{1}{C}}{s + \frac{1}{RC}} \tag{4}$$

which is the same as the result obtained by impedance methods.

Lynn then tried to apply this method to more complex situations. She readily derived the formula

$$\mathcal{L}\left[\frac{d^2x(t)}{dt}\right] = s\mathcal{L}\left[\frac{dx(t)}{dt}\right] - \left.\frac{dx(t)}{dt}\right|_{t=0}$$
$$= s^2\mathcal{L}[x(t)] - sx(0) - \left.\frac{dx(t)}{dt}\right|_{t=0} \tag{5}$$

which can obviously be extended to derivatives of any order. However, when she tried to use (2) and (5) to transform the second-order equation of Example 3.4–1, relating the input $i(t)$ and the output $v(t)$ for the circuit shown to the right,



$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{1}{C}\left(\frac{di(t)}{dt} + \frac{R}{L}i(t)\right) \tag{6}$$

she obtained

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)V(s) - \left(s + \frac{R}{L}\right)v(0) - \left.\frac{dv(t)}{dt}\right|_{t=0} = \frac{1}{C}\left(s + \frac{R}{L}\right)I(s) - \frac{1}{C}i(0). \tag{7}$$

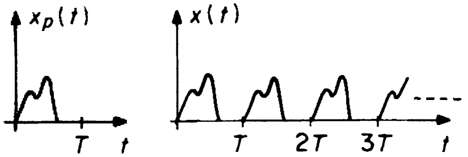
Now setting $v(0) = dv(t)/dt|_{t=0} = 0$ does not appear to give the same result as impedance methods for the ZSR system function $H(s) = V(s)/I(s)$ unless $i(0) = 0$.

Lynn was baffled and consulted her roommate Anna Logg. Anna, whose approach to problems was more physical than mathematical, took one look at the circuits and said "Oh, the trouble is that setting $v(t)$ and $dv(t)/dt$ to zero in the second circuit does not necessarily imply that the circuit is in the zero state." Show that Anna is right and explain how this accounts for Lynn's difficulties in interpreting equation (7). Show in general that a necessary and sufficient condition such that setting the output and its first $N - 1$ derivatives to zero forces an N^{th}-order system to be in the zero state is that the input-output system function have no zeros for finite values of s.

Problem 3.4

Most systems, linear or not, eventually yield a periodic response to a suddenly applied periodic input. Unless the system happens to be in exactly the right state at the time of input application, however, there will be a nonzero interval at the start during which "transients" die out. A classical method of calculating the ultimate periodic response in such cases is to treat the state at some moment after the periodic response has been established as an algebraic unknown, compute the response of the system to the next complete period of the stimulus in terms of this unknown initial state, equate the state at the end of this period to the initial state, and solve the resulting equations for the required initial state. An alternative scheme for LTI systems uses the $\mathcal{L}$-transform and insight into the system response structure as illustrated below.

a) Let $x(t)$ be constructed by repeating periodically with period T a pulse waveform $x_p(t)$ of duration $\leq T$ as indicated in the figure to the right. Show that

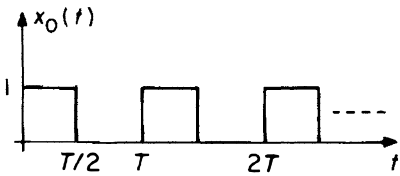


$$\mathcal{L}[x(t)] = X(s) = \frac{X_p(s)}{1-e^{-sT}}, \quad \text{where} \quad X_p(s) = \mathcal{L}[x_p(t)].$$

(HINT: Argue that, for $\Re e[s] > 0$,

$$\frac{1}{1-e^{-sT}} = 1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots.)$$

b) As a specific example, find the $\mathcal{L}$-transform of the waveform $x_0(t)$ shown to the right.



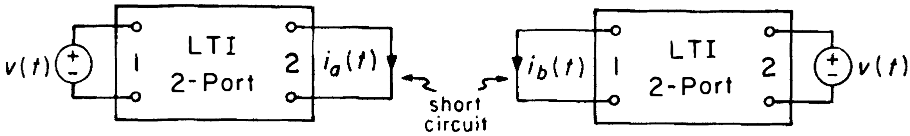
c) Suppose that $x_0(t)$ from (b) is the input to an LTI system with system function $H(s) = 1/(s+\alpha)$, $\alpha > 0$. Sketch the pole locations in the s-plane of the ZSR, $Y(s) = H(s)X(s)$. (Pay particular attention to the possibility of poles located along $s = j\omega$.)

d) The poles in (c) can be divided into two classes—a finite number of poles located inside the half-plane $\Re e[s] < 0$, and an infinite number of poles located along the $j\omega$-axis. The former correspond to transient terms in $y(t)$ that decay with time; the latter correspond to continuing sinusoids that superimpose to comprise the periodic part of the response. Find the transform of the periodic part of the ZSR in (c) by subtracting away from $Y(s)$ those terms in the partial-fraction expansion of $Y(s)$ that correspond to the left-half-plane poles.

e) Rearrange the result in (d) so it has the form derived in (a) and thus identify the transform $Y_p(s)$ of one period of $y(t)$. Do an inverse transform to obtain $y_p(t)$ and sketch your result.

Problem 3.5

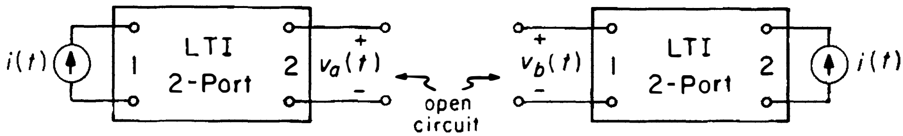
Physically, an LTI 2-port is said to be *reciprocal** if the ZSR current in a short across one pair of terminals in response to an arbitrary voltage source across the other pair of terminals is independent of which end is shorted and which driven—as implied in the figure below. The 2-port is reciprocal if $i_a(t) = i_b(t)$ (ZSR) for all $v(t)$.



a) If the 2-port is described by the short-circuit admittance equations of the appendix to this chapter, show that a necessary and sufficient condition for reciprocity is

$$Y_{12}(s) = Y_{21}(s).$$

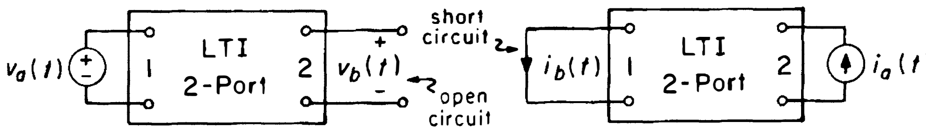
b) Show that an equivalent test for reciprocity is that the voltages $v_a(t)$ and $v_b(t)$ be the same under the two conditions illustrated below for an arbitrary current $i(t)$.



c) Show that another equivalent test for reciprocity is that

$$\frac{V_b(s)}{V_a(s)} = \frac{I_b(s)}{I_a(s)}$$

where the voltages and currents are defined by the circuits illustrated below.

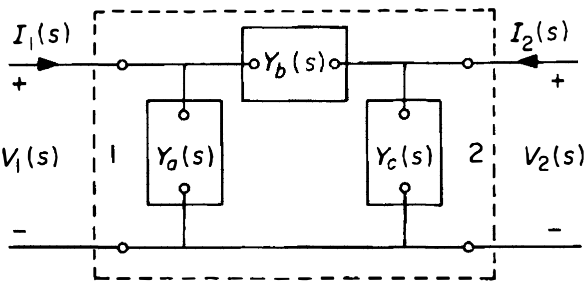


*It can be shown that a sufficient condition for a 2-port to be reciprocal is that it contain only LTI R's, L's, C's, and coupled coils (e.g., ideal transformers). This is sometimes called the *Network Reciprocity Theorem.* It is straightforward to show that this Reciprocity Theorem is a consequence of the symmetry of the node equations as they are usually written for such circuits; that is, the term in the KCL equation at node i proportional to the voltage at node j is identical to the term in the KCL equation at node j proportional to the voltage at node i (see, e.g., E. A. Guillemin, *Introductory Circuit Theory* (New York, NY: John Wiley, 1953) p. 148ff). A more elegant but rather less transparent argument follows from Tellegen's Theorem—see Problem 4.4 and, e.g., C. A. Desoer and E. S. Kuh, *Basic Circuit Theory* (New York, NY: McGraw-Hill, 1969) p. 681ff. The reciprocity concept is readily extended to n-ports and even to non-linear circuits. A key theorem due to Brayton is that a network composed of interconnected reciprocal subnetworks is reciprocal (see, e.g., G. F. Oster, A. S. Perelson, and A. Katchalsky, *Quart. Rev. Biophysics*, 6 (1973): 1–138).

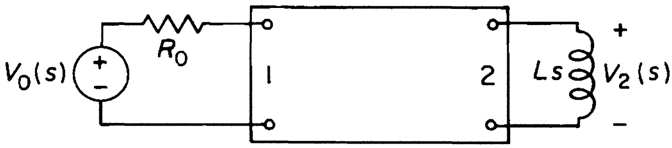
Problem 3.6

a) Show that any reciprocal LTI 2-port can be represented by the *equivalent* Π-*circuit** shown below where the 1-port admittances $Y_a(s)$, $Y_b(s)$, and $Y_c(s)$ are defined in terms of the short-circuit admittances of the 2-port by

$$Y_a(s) = Y_{11}(s) + Y_{12}(s)$$
$$Y_b(s) = -Y_{12}(s) = -Y_{21}(s)$$
$$Y_c(s) = Y_{22} + Y_{21}(s).$$



b) Use the equivalent circuit above and simple techniques of resistive network theory (e.g., Thévenin-Norton equivalences, voltage-divider formulas, etc.) to derive the transfer ratio $V_2(s)/V_0(s)$ in the following diagram and show that the result agrees with the formula derived in the appendix to this chapter.



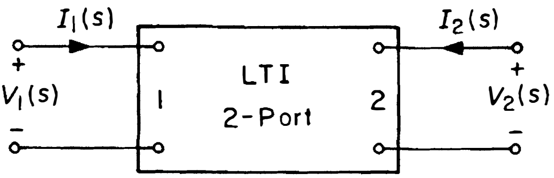
*It must be understood that this is only a mathematically equivalent circuit; there is no guarantee that three 1-port circuits having the admittances $Y_a(s)$, $Y_b(s)$, and $Y_c(s)$ could necessarily actually be constructed out of positive R's, L's, and C's, etc. Moreover, the equivalence holds only for terminal-pair behavior (2-port behavior); for example, the bottom terminals of each pair are shorted together (that is, they are at the same potential) in the equivalent circuit but may not be so connected in the actual 4-terminal network.

Problem 3.7

If the currents rather than the voltages at the ports are taken as the independent variables, then the LTI 2-port shown in the figure below can be characterized by the equations

$$V_1(s) = Z_{11}(s)I_1(s) + Z_{12}(s)I_2(s)$$
$$V_2(s) = Z_{21}(s)I_1(s) + Z_{22}(s)I_2(s)$$

where the $Z_{ij}(s)$ are called the *open-circuit driving-point* and *transfer impedances* of the 2-port.



a) Devise experiments to measure the open-circuit impedances analogous to those described in the appendix to this chapter for the short-circuit admittances.

b) Derive the following expressions for the open-circuit impedances in terms of the short-circuit admittances, and vice versa:

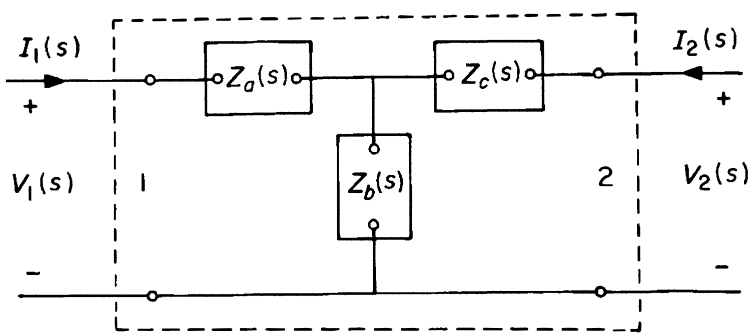
$$Z_{11}(s) = \frac{Y_{22}(s)}{Y_{11}(s)Y_{22}(s) - Y_{12}(s)Y_{21}(s)}; \quad Y_{11}(s) = \frac{Z_{22}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)Z_{21}(s)}$$
$$Z_{12}(s) = \frac{-Y_{21}(s)}{Y_{11}(s)Y_{22}(s) - Y_{12}(s)Y_{21}(s)}; \quad Y_{12}(s) = \frac{-Z_{21}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)Z_{21}(s)}$$
$$Z_{21}(s) = \frac{-Y_{12}(s)}{Y_{11}(s)Y_{22}(s) - Y_{12}(s)Y_{21}(s)}; \quad Y_{21}(s) = \frac{-Z_{12}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)Z_{21}(s)}$$
$$Z_{22}(s) = \frac{Y_{11}(s)}{Y_{11}(s)Y_{22}(s) - Y_{12}(s)Y_{21}(s)}; \quad Y_{22}(s) = \frac{Z_{11}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)Z_{21}(s)}$$

c) Show that the tests for reciprocity described in Problem 3.5 imply and are implied by

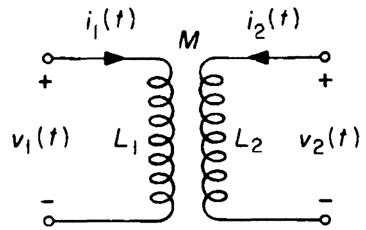
$$Z_{12}(s) = Z_{21}(s).$$

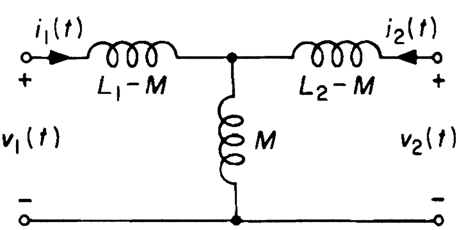
d) Show that any reciprocal LTI 2-port can be represented by the equivalent T-circuit shown below, where the 1-port impedances $Z_a(s)$, $Z_b(s)$, and $Z_c(s)$ are defined in terms of the open-circuit impedances of the 2-port by

$$Z_a(s) = Z_{11}(s) - Z_{12}(s)$$
$$Z_b(s) = Z_{12}(s) = Z_{21}(s)$$
$$Z_c(s) = Z_{22}(s) - Z_{21}(s).$$



e) Open-circuit 2-port impedances are very useful for describing the properties of a pair of *coupled coils*—a set of two coils or inductors arranged physically (e.g., wound on a common core) so that the changing flux generated by a changing current in one coil links both coils and thus induces voltages in both coils. The symbol for a pair of coupled coils is shown below.





The corresponding equations are

$$v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$$
$$v_2(t) = M\frac{di_1(t)}{dt} + L_2\frac{di_2(t)}{dt}.$$

The *mutual inductance* M may have either sign (depending on the choice of reference directions for the currents) and is constrained in magnitude by the fact that physically the *coupling coefficient*, $k = |M|/\sqrt{L_1L_2}$, must be less than unity. Argue that the arrangement of three inductors in a T-circuit as shown to the right above is mathematically equivalent as a 2-port to the pair of coupled coils shown to the left. Could any pair of coupled coils therefore physically be replaced by three uncoupled inductors of appropriate values without altering the behavior of the circuit? Explain.

f) Devise an alternative equivalent circuit for a pair of coupled coils in the form of a Π-circuit of inductors, based on the short-circuit admittance equivalent circuit of Problem 3.6. Give values for the elements in your circuit in terms of L_1, L_2, and M.

Problem 3.8

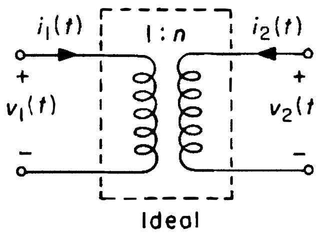
In addition to the open-circuit impedance or short-circuit admittance representations for LTI 2-ports discussed in Problem 3.7 and the appendix to this chapter, four other ways of characterizing LTI 2-port behavior are possible—corresponding to the four remaining ways of picking two independent variables from the four terminal quantities $V_1(s)$, $V_2(s)$, $I_1(s)$, and $I_2(s)$. The preferred choice among the 6 possibilities in a particular practical case usually involves a balance of two factors:

1. For which representation is the description of a particular 2-port simplest?
2. For which representation are the constraints imposed by the external circuits most readily expressed?

This problem explores these factors with several examples.

a) The *ideal transformer* shown to the right is characterized by the equations

$$v_2(t) = nv_1(t)$$
$$ni_2(t) = -i_1(t).$$



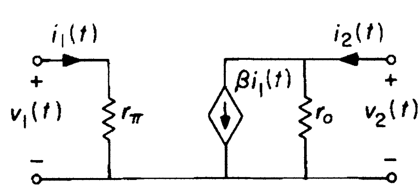
It may be considered the limit of a pair of coupled coils (see Problem 3.7) as $k \to 1$ and $L_1 \to \infty$, $L_2 \to \infty$ with $L_2/L_1 = n^2$. (n is approximately the ratio of the number of turns in the secondary winding (L_2) to the number of turns in the primary winding (L_1)). Show that both the open-circuit impedances and the short-circuit admittances of Problem 3.7 and the appendix to this chapter are infinite for the ideal transformer, but that the so-called *ABCD representation* of a 2-port,

$$V_2(s) = A(s)V_1(s) + B(s)I_1(s)$$
$$I_2(s) = C(s)V_1(s) + D(s)I_1(s)$$

exists. Find $A(s)$, $B(s)$, $C(s)$, and $D(s)$ for the ideal transformer.

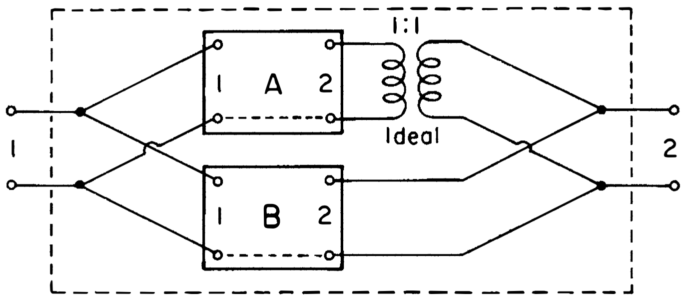
b) A simplified incremental circuit for a transistor is shown to the right. Find the parameter values corresponding to this circuit for the *hybrid representation* of a 2-port,

$$V_1(s) = H_{11}(s)I_1(s) + H_{12}(s)V_2(s)$$
$$I_2(s) = H_{21}(s)I_1(s) + H_{22}(s)V_2(s).$$



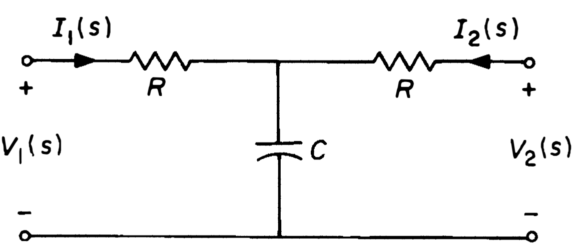
c) Determine the condition that must be satisfied by the $H_{ij}(s)$ parameters of (b) if the 2-port described by this representation is reciprocal. Is the incremental circuit for the transistor reciprocal?

d) Suppose two 2-ports are connected in parallel to form the single 2-port represented by the dashed box in the figure below. (The purpose of the 1:1 ideal transformer is to ensure that the 2-port conditions remain satisfied for each 2-port in the parallel combination. The transformer is unnecessary if, for example, both constituent 2-ports have common grounds connecting their lower terminals as shown by the dashed lines.) Find the short-circuit admittances characterizing the parallel connection in terms of the short-circuit admittances of each constituent 2-port.



e) Draw a diagram showing how two 2-ports should be interconnected to form a new 2-port such that the open-circuit driving-point and transfer impedances of the interconnection are the sums of the corresponding open-circuit driving-point and transfer impedances of the constituent 2-ports.

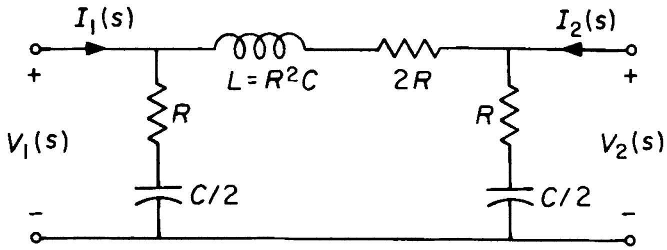
Problem 3.9



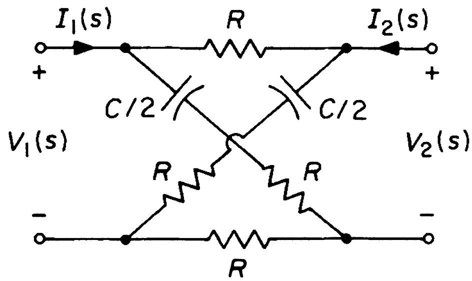
a) For the circuit above, find the short-circuit driving-point and transfer admittances in the representation

$$I_1(s) = Y_{11}(s)V_1(s) + Y_{12}(s)V_2(s)$$
$$I_2(s) = Y_{21}(s)V_1(s) + Y_{22}(s)V_2(s).$$

b) Show that the circuit on the next page is equivalent to that above, in the sense that it has the same 2-port representation.



c) Show that the circuit below is also equivalent to those above as a 2-port.

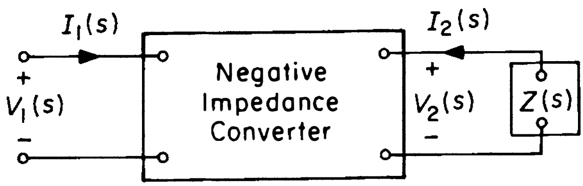


d) Is there any set of measurements that could be made at the ports of the three circuits above to distinguish one from the others?

e) Is there any set of measurements that could be made at the terminals of the three circuits above to distinguish one from the others?

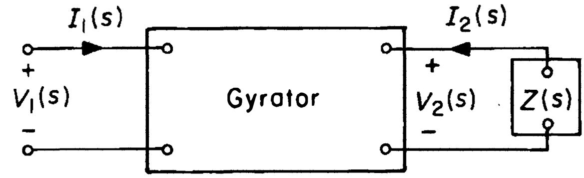
Problem 3.10

a) A *negative-impedance converter* is a 2-port device that converts an impedance at its output to appear at the input as the *negative* of that impedance, as shown below.



$$Z_{in}(s) = \frac{V_1(s)}{I_1(s)} = -Z(s)$$

A *gyrator* is a 2-port device that converts an impedance at its output to appear at the input as the *reciprocal* of that impedance, as shown below.



$$Z_{in}(s) = \frac{V_1(s)}{I_1(s)} = \frac{R^2}{Z(s)}$$

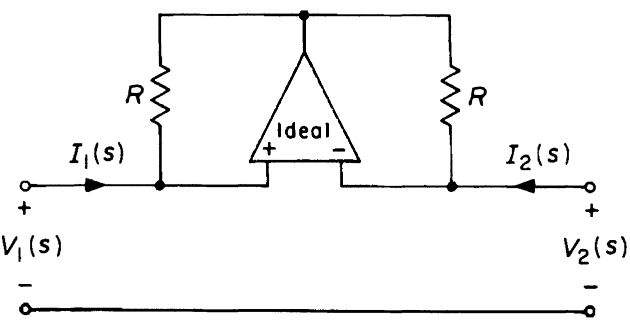
Describe the similarities and differences in the input impedances to these two devices if $Z(s)$ is a pure capacitor, $Z(s) = 1/Cs$. (Consider in particular the behavior for $s = j\omega$.) Which input impedance, if either, is indistinguishable from the impedance of an inductor?

b) Both the negative-impedance converter and the gyrator can be described by appropriate choices of the parameters in the $ABCD$ representation of a 2-port described in Problem 3.8:

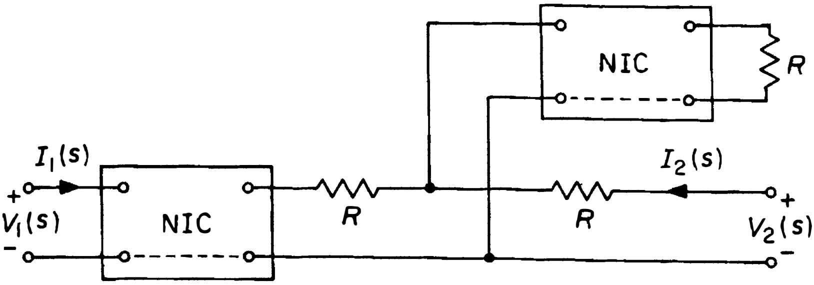
$$V_2(s) = A(s)V_1(s) + B(s)I_1(s)$$
$$I_2(s) = C(s)V_1(s) + D(s)I_1(s).$$

Determine the values of the parameters that describe each device (the answers may not be unique). Are these devices reciprocal (see Problem 3.5)?

c) Show that the circuit below behaves as a negative-impedance converter.



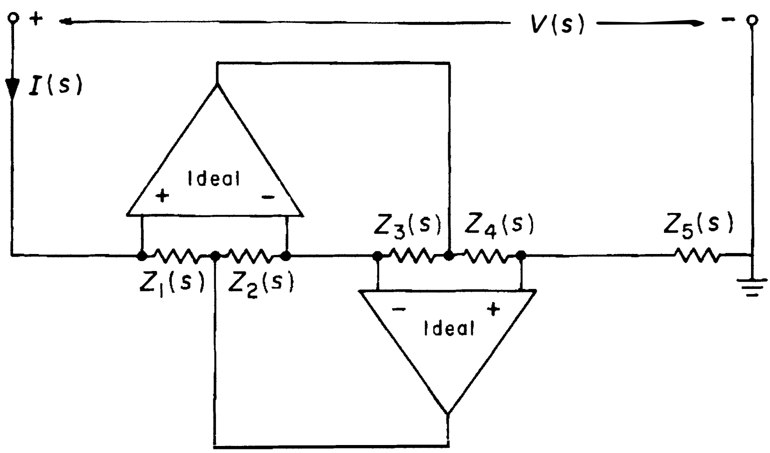
d) Show that the circuit on the next page behaves as a gyrator. "NIC" is a negative-impedance converter as in (c).



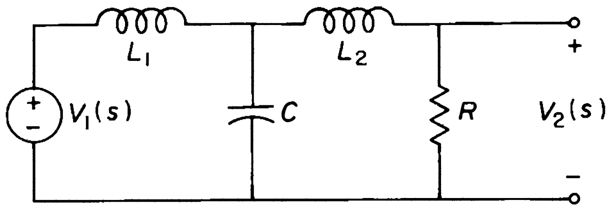
Problem 3.11

a) Another circuit that can be used to realize a gyrator (see Problem 3.10) is as shown below. Argue that

$$Z(s) = \frac{V(s)}{I(s)} = \frac{Z_1(s)Z_3(s)Z_5(s)}{Z_2(s)Z_4(s)}.$$



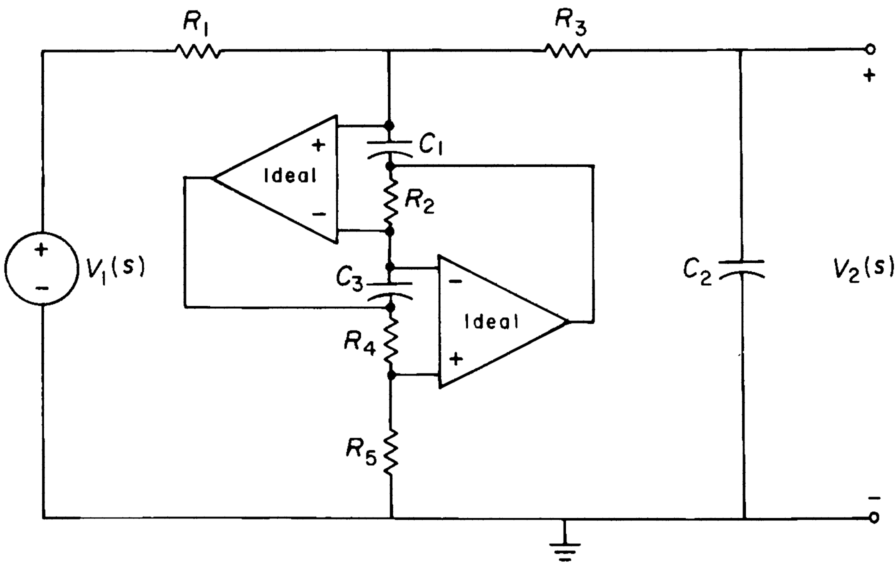
b) If $Z_4(s)$ is a capacitor and the remaining impedances are resistors, the circuit above behaves at its terminals as an inductor of value $L = \frac{R_1R_3R_5C_4}{R_2}$. The circuit can thus be used to replace an inductor in a filter design, provided that the inductor has one terminal grounded. Unfortunately, lowpass filters, such as the Butterworth filter shown on the next page, have inductors with both terminals above ground.



Butterworth lowpass filter, cutoff frequency = ω_0 rad/sec.

$$L_1 = \frac{3R}{2\omega_0}, \quad L_2 = \frac{R}{2\omega_0}, \quad C = \frac{4}{3R\omega_0}$$

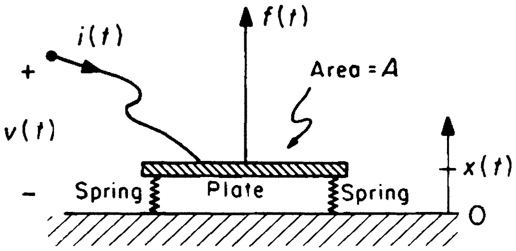
One way of using the circuit in (a) to realize such a filter is to observe that if all of the impedances in any circuit are divided by ks, then all voltage ratios in that circuit remain unchanged. (Convince yourself that this statement is true.) However, inductors get converted into resistors of value L/k, resistors get converted into capacitors of value k/R, and capacitors get converted into "double capacitors" with impedances $1/kCs^2$. A "double capacitor" can be realized with the circuit in (a) if $Z_1(s)$ and $Z_3(s)$ are capacitors. Use these ideas to find values for the elements in the circuit below that will realize a Butterworth lowpass filter with cutoff frequency equal to 1000 Hz. Try to keep all resistor values in the range 10 kΩ to 100 kΩ.



Problem 3.12

A *transducer* is a system for transforming energy from one form to another. Frequently, one of the forms is electrical; examples are an almost endless variety of mechano-electric devices such as motors, generators, loudspeakers, phonograph pickups, and accelerometers, as well as a host of thermo-electric, chemo-electric, optico-electric and other devices. The important dynamic behavior of a transducer often corresponds to a situation in which the perturbations in the variables about some steady condition are small enough that the system can be described incrementally as a linear 2-port. In addition, the energy transduction efficiency of many transducers is high enough that the device can be modelled as nearly lossless. In these cases, the linear 2-port must satisfy an interesting reciprocity condition, as this problem will illustrate.

a) The diagram below shows a simple mechano-electric transducer constructed from a massless plate of area A supported a variable distance $x(t)$ away from a fixed ground plane by electrically insulated springs. A mechanical force $f(t)$ can be applied to the plate. The plate can also be charged electrically through a flexible wire, forming a capacitor with the ground plane. This arrangement describes the essential features of a variety of useful devices such as condenser microphones and force transducers.



The stored energy in this system is

$$E[q(t), x(t)] = \frac{q^2(t)x(t)}{2\epsilon_0 A} + \frac{K(x(t) - x_0)^2}{2}$$

where

$q(t) =$ the electrical charge on the movable plate,

$\epsilon_0 =$ the permittivity of free space,

$x_0 =$ the resting length of the springs,

$K =$ the effective combined stiffness of the springs.

Since the system is lossless, electrical or mechanical work done on the system yields a corresponding increase in stored energy. Incrementally,

$$\Delta E[q(t), x(t)] = v(t)\Delta q(t) + f(t)\Delta x(t).$$

Thus, it must follow that

$$v(t) = \frac{\partial E[q(t), x(t)]}{\partial q(t)}, \qquad f(t) = \frac{\partial E[q(t), x(t)]}{\partial x(t)}.$$

Find formulas for $v(t)$ and $f(t)$ in terms of $q(t)$ and $x(t)$.*

b) Assume that each of the variables can be described as a large constant (quiescent) value plus a small perturbation:

$$\begin{aligned} f(t) &= f_{00} + f_i(t), \quad & |f_{00}| \gg |f_i(t)| \\ v(t) &= v_{00} + v_i(t), \quad & |v_{00}| \gg |v_i(t)| \\ x(t) &= x_{00} + x_i(t), \quad & |x_{00}| \gg |x_i(t)| \\ q(t) &= q_{00} + q_i(t), \quad & |q_{00}| \gg |q_i(t)|. \end{aligned}$$

Let $F_i(s)$, $V_i(s)$, $U_i(s)$, and $I_i(s)$ stand respectively for the $\mathcal{L}$-transforms of $f_i(t)$, $v_i(t)$, $dx_i(t)/dt$, and $dq_i(t)/dt$. Show that these quantities are related by the following 2-port equations:

$$\begin{aligned} F_i(s) &= \frac{K}{s} U_i(s) + \frac{q_{00}}{\epsilon_0 A s} I_i(s) \\ V_i(s) &= \frac{q_{00}}{\epsilon_0 A s} U_i(s) + \frac{x_{00}}{\epsilon_0 A s} I_i(s). \end{aligned}$$

c) Note that the *cross* or *coupling* terms in the 2-port equations in (b) (i.e., the term in the force equation proportional to the current, and the term in the voltage equation proportional to the velocity) have identical coefficients. This is a reciprocity condition of exactly the same sort discussed in Problem 3.5 for purely electrical circuits. Show that such a reciprocity condition holds for any system in which the stored energy can be written as a function of the displacement $x(t)$ at the mechanical terminal and the charge $q(t)$ at the electrical terminal.† HINT: Make use of the mathematical fact that

$$\frac{\partial}{\partial q(t)}\left(\frac{\partial E[q(t), x(t)]}{\partial x(t)}\right) = \frac{\partial}{\partial x(t)}\left(\frac{\partial E[q(t), x(t)]}{\partial q(t)}\right).$$

*The internal energy written in terms of generalized "displacements," such as $x(t)$ and $q(t)$, is called in physics the *Hamiltonian* of the system. The partial derivative of the Hamiltonian with respect to a particular "displacement" yields the associated generalized "force." If the system contains stored magnetic and kinetic energy, then the corresponding "displacements" are the magnetic flux and the mechanical momentum, and the "forces" are the electric current and the mechanical velocity, respectively.

†A reciprocity condition of this kind is called a *Maxwell relation* in thermodynamics.