4

POLES AND ZEROS

4.0 Introduction

The system function $H(s)$ for any lumped LTI circuit always has the form of a ratio of polynomials in s, that is, a rational function. By the Fundamental Theorem of Algebra, any polynomial can be factored in terms of its roots,

$$a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 \\ = a_n(s - s_1)(s - s_2)\cdots(s - s_n).$$

Thus a rational system function $H(s)$ can always be written in the form

$$H(s) = K\frac{(s - s_{z1})(s - s_{z2})\cdots(s - s_{zN})}{(s - s_{p1})(s - s_{p2})\cdots(s - s_{pM})}$$

and is completely specified (except for a real multiplicative constant) by the roots of the numerator and denominator polynomials—the *zeros*, s_{zi}, and the *poles*, s_{pi}, of $H(s)$. The type of system function (e.g., driving-point impedance or transfer ratio), the types of elements that compose the circuit, and the topology of the network all place constraints on zero and pole locations. On the other hand, once the pole and zero locations are specified, much can be said—qualitatively as well as quantitatively—about the behavior of the system, including in particular the general trends of its frequency response $H(j\omega)$. These are the topics of this chapter. With these topics we shall essentially complete our study of circuits as interconnections of the most primitive electrical elements. In the next chapter we shall start to explore the behavior of systems whose building blocks are themselves systems or circuits of some complexity.

4.1 Pole-Zero Diagrams

Often the most convenient and evocative way to summarize the characterization of a system function is to plot its pole and zero locations graphically in the complex s-plane; this is called a *pole-zero diagram*. An example is shown in Figure 4.1–1.

The type of elements used as well as the network structure constrain the regions of the s-plane in which the poles and zeros may lie. First, it is important

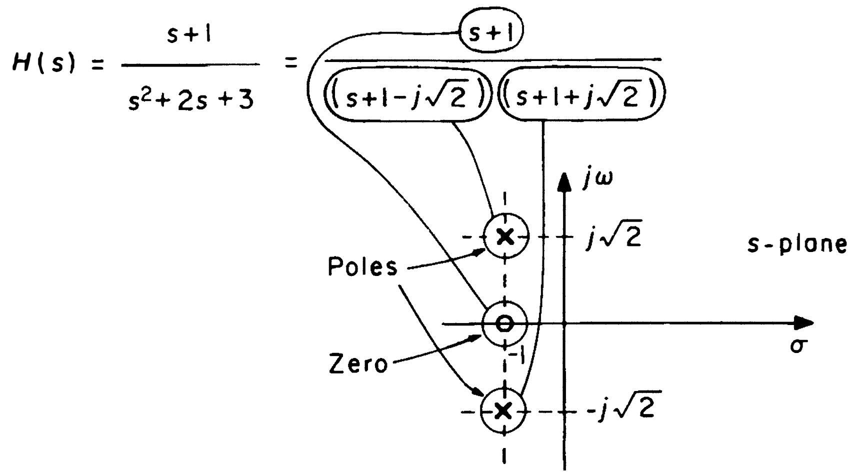

Figure 4.1-1. An example of a pole-zero diagram.

to note (as illustrated in Chapter 2) that the fact that the element values are real numbers forces the pole and zero locations either to lie along the real σ-axis (e.g., the zero in Figure 4.1-1) or to occur in conjugate complex pairs at mirror-image points with respect to the σ-axis (e.g., the poles in Figure 4.1-1). Moreover, if the network is composed of positive R's, L's, and C's only (coupled coils and transformers are allowed, but no controlled sources), then the poles of the system function must lie in the left half-s-plane or on the $j\omega$-axis (i.e., $\sigma \leq 0$); see Problem 4.4 for a proof. Networks composed entirely of positive R's, L's, and C's are said to be *passive*. System functions whose poles do not lie in the right half-s-plane are called *stable*; a passive LTI network is stable.*

For special classes of passive circuits the locations of the poles are even more restricted (see Problem 4.4 for a further discussion):

1. For LTI networks composed of positive R's and L's only, or positive R's and C's only (no controlled sources), the poles must all lie on the negative real axis. RC or RL networks are sometimes called *relaxation networks* because their ZIR is a weighted sum of monotonically decaying exponentials.
2. For LTI networks constructed from positive L's and C's only (no controlled sources), the poles must all lie on the $j\omega$ axis. Idealized LC networks are called *lossless networks* and are marginally stable; their ZIR is a weighted sum of undamped sinusoids that neither grow nor decay.

*The converse is not true; a stable network need not be passive, as we have already seen in various op-amp examples and as we shall discuss in detail in Chapter 6. More generally, passivity implies that the energy one can extract from an element or circuit cannot exceed a finite bound determined by the initial state—see, e.g., J. L. Wyatt, Jr., et al., *IEEE Trans. Cir. & Sys.*, *CAS-28* (1981): 48–61. Stability implies that the effects of small perturbations remain small; an LTI system is clearly unstable if its ZIR contains growing exponentials—if the poles of the system function lie in the right half-s-plane—because then any disturbance, no matter how small, will ultimately yield a large effect. For a further discussion of stability, including more careful definitions, see Chapter 6.

The zeros of a passive system function—unlike the poles—can in general lie anywhere in the complex plane, although again special system types impose restrictions:

1. If $H(s)$ is a driving-point impedance, that is, if

$$V(s) = H(s)I(s)$$

where $V(s)$ is the $\mathcal{L}$-transform of the ZSR voltage across the impedance $H(s)$ induced by a current source $I(s)$, then (as pointed out in Section 3.2) $1/H(s)$ is also a system function, namely a driving-point admittance satisfying the equation

$$I(s) = \frac{1}{H(s)}V(s)$$

where $I(s)$ is the $\mathcal{L}$-transform of the ZSR current flowing through the impedance $H(s)$ in response to a voltage source $V(s)$. In this case the zeros of $H(s)$, which are the poles of $1/H(s)$, must satisfy the same types of constraints as the poles: In general they must lie in the left half-plane for passive *RLC* networks, along the negative σ-axis for passive *RC* and *RL* networks, or along the $j\omega$-axis for passive *LC* networks.*

2. A *ladder network* is a circuit that looks topologically like a "ladder" with alternating series and shunt branches. It can be shown that the zeros of any transfer function of a ladder network must lie in the left half-plane. For an *RC* or *RL* ladder network the zeros must lie on the negative σ-axis; for an *LC* ladder network they must lie on the $j\omega$-axis.

Example 4.1–1

The *Twin-T network* shown in Figure 4.1–2 is an important and useful *RC* circuit.

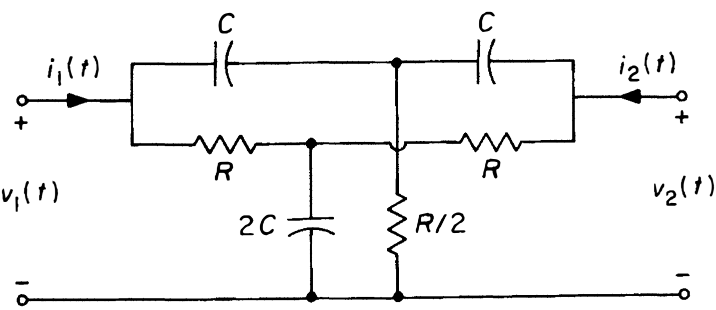

Figure 4.1–2. A Twin-T network.

*These conditions on the zeros are necessary but not sufficient for $H(s)$ to be a driving-point impedance. Thus for *RC*, *RL*, or *LC* driving-point impedances, the poles and zeros must in fact alternate, as can be shown by extension of the methods of Problems 4.5 and 4.6. For the general case of an *RLC* driving-point impedance, restrictions on the zeros can be derived from the fact that the impedance must be a "positive-real function" as discussed in Problem 4.5.

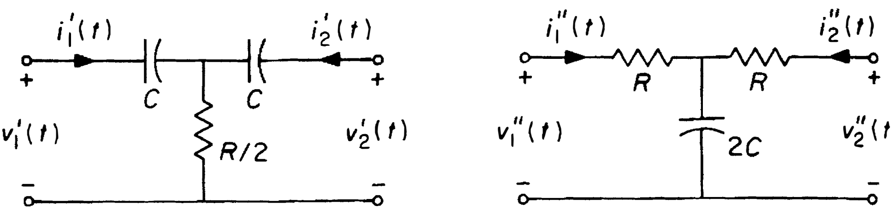

Figure 4.1–3. Component T-networks from which the Twin-T of Figure 4.1–2 can be constructed.

The short-circuit driving-point and transfer admittances of the 2-port shown in Figure 4.1–2 (see the appendix to Chapter 3) can easily be found as the sums of the short-circuit admittances of the two simple ladder networks (called *T-networks* because of their topology) that in parallel comprise it, as shown in Figure 4.1–3:

$$Y'_{11}(s) = Y'_{22}(s) = \frac{Cs(s+2/RC)}{2(s+1/RC)} \qquad Y''_{11}(s) = Y''_{22}(s) = \frac{s+1/2RC}{R(s+1/RC)}$$

$$Y'_{12}(s) = Y'_{21}(s) = \frac{-Cs^2}{2(s+1/RC)} \qquad Y''_{12}(s) = Y''_{21}(s) = \frac{-1}{2R^2C(s+1/RC)}.$$

Hence

$$Y_{11}(s) = Y_{22}(s) = \frac{Cs(s+2/RC)}{2(s+1/RC)} + \frac{s+1/2RC}{R(s+1/RC)} = \frac{C\left(s^2+4s/RC+1/R^2C^2\right)}{2(s+1/RC)}$$

and

$$Y_{12}(s) = Y_{21}(s) = \frac{-Cs^2}{2(s+1/RC)} - \frac{1}{2R^2C(s+1/RC)} = \frac{-C\left(s^2+1/R^2C^2\right)}{2(s+1/RC)}.$$

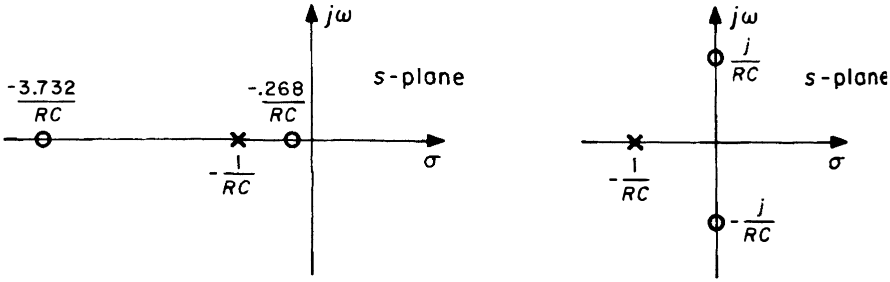

Figure 4.1–4. Pole-zero diagrams for $Y_{11}(s)$ (left) and $Y_{12}(s)$ (right).

We note that $Y_{11}(s)$ is the driving-point admittance of an RC circuit—both poles and zeros (as shown in Figure 4.1–4) are on the negative real axis (and alternate as required by the footnote on page 107). The same properties also characterize $Y'_{11}(s)$ and $Y''_{11}(s)$. On the other hand, $Y_{12}(s)$ is the transfer admittance of an RC circuit—the pole is on the negative real axis as required, but the zeros are on the $j\omega$-axis. (The Twin-T is not a ladder network; thus the zeros of the transfer admittance need not lie on the negative real axis. Note, however, that $Y'_{12}(s)$ and $Y''_{12}(s)$, which are the transfer admittances of RC ladder networks, have their zeros at $s = 0$ or $s = \infty$, which do lie on the negative real axis, at least in a limiting sense.) Because it has a zeros on the $j\omega$-axis, the Twin-T network can be used to block or "trap" input sinusoids with frequencies near $\omega = 1/RC$, preventing such input components from appearing at the output. (For an application see Problem 4.13.)

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4.2 Vectorial Interpretation of $H(j\omega)$

Many of the important features of $H(j\omega)$ as a function of $j\omega$ can be deduced directly from the pole-zero diagram of $H(s)$. A basis for this use of the pole-zero diagram is the observation that the complex number $s - s_0$, represented as a vector in the complex plane, is the vector joining the tips of the vectors s and s_0. The length of this vector is $|s - s_0|$; its angle is $\angle(s - s_0)$. These ideas are illustrated in Figure 4.2–1.

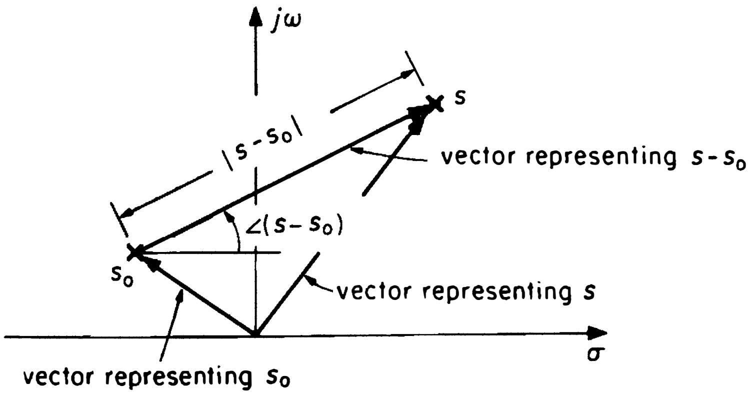

Figure 4.2–1. Vectorial interpretation of the complex numbers s, s_0, and $s - s_0$.

Suppose, now, that we write

$$H(s) = K\frac{(s - s_{z1})(s - s_{z2})\cdots(s - s_{zN})}{(s - s_{p1})(s - s_{p2})\cdots(s - s_{pM})}. \tag{4.2–1}$$

Then, expressing each term in magnitude-angle form and setting $s = j\omega$, we have

$$|H(j\omega)| = K\frac{\prod_{i=1}^{N} |j\omega - s_{zi}|}{\prod_{i=1}^{M} |j\omega - s_{pi}|} \tag{4.2–2}$$

$$\angle H(j\omega) = \sum_{i=1}^{N} \angle(j\omega - s_{zi}) - \sum_{i=1}^{M} \angle(j\omega - s_{pi}). \tag{4.2–3}$$

That is, $|H(j\omega)|$ is K times the ratio of the product of the lengths of the vectors from each zero to the point $j\omega$ on the $j\omega$-axis divided by the product of the lengths of the vectors from each pole to the point $j\omega$. Similarly, the angle of $H(j\omega)$ is the difference of the sums of the individual term angles. The usefulness of this vectorial interpretation is best illustrated by an example.

Example 4.2–1

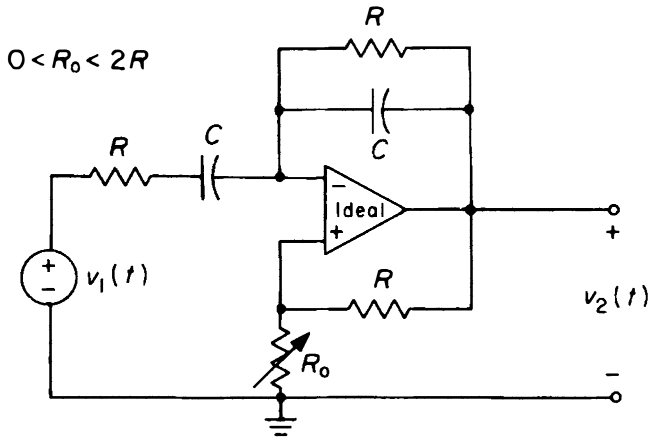

Figure 4.2–2. Circuit for Example 4.2–1.

Let us find the system function of the op-amp circuit shown in Figure 4.2–2. Employing the usual ideal op-amp approximations and using superposition and voltage-divider rules, we can readily find a relation between $V_1(s)$ and $V_2(s)$ by equating the voltages at the inverting and noninverting terminals:

$$\frac{V_2(s)\left(R + \frac{1}{Cs}\right)}{R + \frac{1}{Cs} + \frac{R(1/Cs)}{R+1/Cs}} + \frac{V_1(s)\frac{R(1/Cs)}{R+1/Cs}}{R + \frac{1}{Cs} + \frac{R(1/Cs)}{R+1/Cs}} = V_2(s)\frac{R_0}{R_0 + R}. \tag{4.2–4}$$

Rearranging (4.2–4) gives

$$\frac{V_2(s)}{V_1(s)} = H(s) = K\frac{\frac{1}{Q}\left[\frac{s}{\omega_0}\right]}{\left[\frac{s}{\omega_0}\right]^2 + \frac{1}{Q}\left[\frac{s}{\omega_0}\right] + 1} \tag{4.2–5}$$

where

$$\omega_0 = \frac{1}{RC} \tag{4.2–6}$$

$$Q = \frac{R}{2R - R_0} \quad \left(\text{so that } \frac{1}{2} < Q < \infty \text{ if } 0 < R_0 < 2R\right) \tag{4.2–7}$$

$$K = -\frac{R_0 + R}{2R - R_0} = H(j\omega_0). \tag{4.2–8}$$

The pole-zero diagram for $H(s)$ is shown in Figure 4.2–3; as R_0 is changed, the poles move along a circle of radius ω_0. When $Q = 1/2$ ($R_0 = 0$), the poles are coincident on the negative σ-axis at $\sigma = -\omega_0 = -1/RC$. When $Q \to \infty$ ($R_0 \to 2R$), the poles approach the points $\pm j\omega_0$ (and indeed if $R_0 > 2R$, the poles move into the right half-plane, which implies an unstable system).

The pole-zero diagram for this circuit as R_0 is adjusted behaves in precisely the same way as the pole-zero diagram for the impedance $Z(s)$ of the parallel resonant circuit shown in Figure 4.2–4 as the shunt resistance R is varied:

$$Z(s) = \frac{1}{\frac{1}{R} + \frac{1}{Ls} + Cs} = R\frac{\frac{1}{Q}\left[\frac{s}{\omega_0}\right]}{\left[\frac{s}{\omega_0}\right]^2 + \frac{1}{Q}\left[\frac{s}{\omega_0}\right] + 1}$$

where

$$K = R, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad Q = R\sqrt{\frac{C}{L}} = \omega_0 RC = \frac{R}{\omega_0 L}.$$

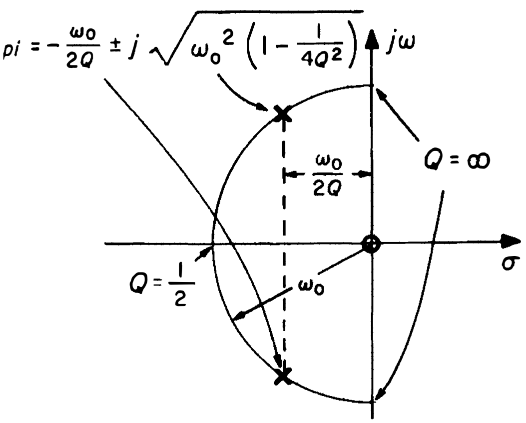

Figure 4.2–3. Poles and zeros for (4.2–5).

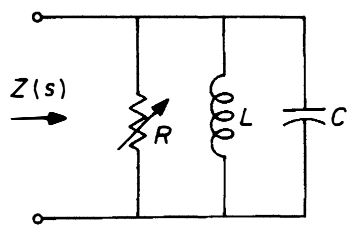

Figure 4.2–4. Parallel resonant circuit.

Thus, if we study the relationship between the pole-zero locations and the frequency response for the op-amp circuit, the results will apply widely to a large class of resonant systems.

Resonance is an interesting phenomenon primarily when Q is large, say 10 or higher. The poles are then relatively close to the $j\omega$-axis, so that the pole-zero diagram becomes approximately as shown to the left in Figure 4.2–5.

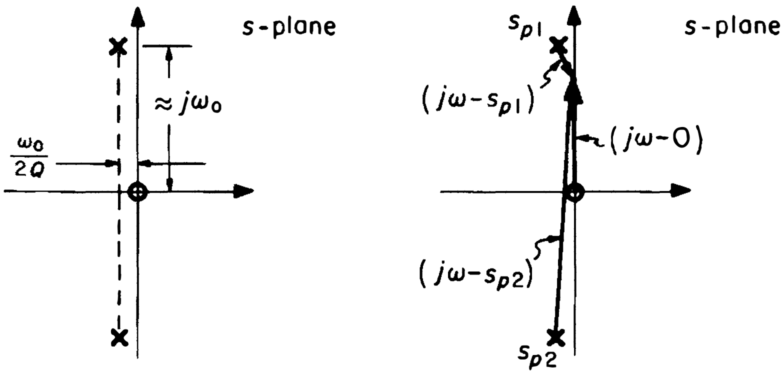

Figure 4.2–5. Pole-zero diagram for a resonant system with large Q.

The frequency response $H(j\omega)$ in this case has the form

$$H(j\omega) = \frac{\frac{K\omega_0}{Q} j\omega}{(j\omega)^2 + \frac{\omega_0}{Q}(j\omega) + \omega_0^2} = \frac{\frac{K\omega_0}{Q}(j\omega - 0)}{(j\omega - s_{p1})(j\omega - s_{p2})}$$

where

$$s_{p1} = s_{p2}^* = \frac{-\omega_0}{2Q} + j\sqrt{\omega_0^2\left(1 - \frac{1}{4Q^2}\right)} \approx \frac{-\omega_0}{2Q} + j\omega_0.$$

To approximate $H(j\omega)$ near $\omega = \omega_0$, we can consider the vectors $(j\omega - 0)$ and $(j\omega - s_{p2})$ to be essentially constant, independent of ω, at the values $j\omega_0$ and $2j\omega_0$, respectively. The behavior of $H(j\omega)$ is then dominated by the rapid change in the vector $(j\omega - s_{p1})$ for ω near ω_0:

$$H(j\omega) \approx \frac{\frac{K\omega_0}{Q} j\omega_0}{\left(j\omega + \frac{\omega_0}{2Q} - j\omega_0\right)2j\omega_0} = \frac{H(j\omega_0)}{1 + j2Q\left(\frac{\omega - \omega_0}{\omega_0}\right)}.$$

At $(\omega - \omega_0)/\omega_0 = \pm 1/2Q$, or $\omega = \omega_0 \pm \omega_0/2Q$, we have $H(j\omega) = e^{\mp j\pi/4} H(j\omega_0)/\sqrt{2}$. These are the *half-power points*, the frequencies at which $|H(j\omega)|^2 = |H(j\omega_0)|^2/2$. Using the approximate formula above, we can sketch a *universal resonance curve* for ω in the vicinity of ω_0 (or for $f = \omega/2\pi$ in the vicinity of $f_0 = \omega_0/2\pi$) as shown in Figure 4.2-6. Note that Q is approximately the center frequency f_0 divided by the half-power bandwidth Δf.

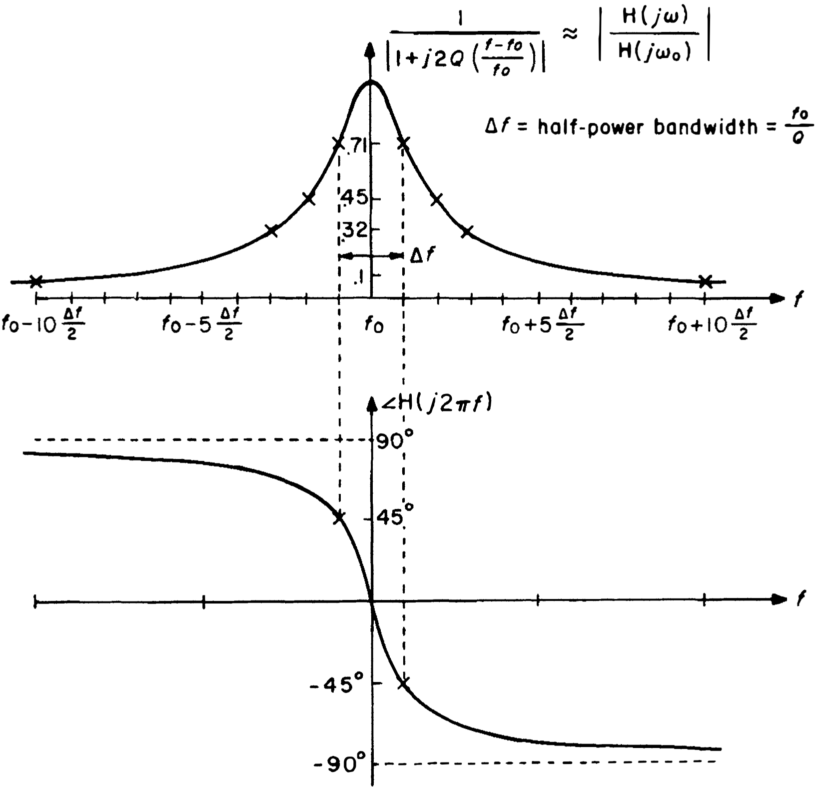

Figure 4.2–6. Universal resonance curve.

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The vectorial interpretation of $H(s)$ shows that for *any* $H(s)$ having a pole close to the $j\omega$-axis, the universal resonance curve describes $H(j\omega)$ for values of ω near enough to the pole so that changes in the vectors from $j\omega$ to all the other poles and zeros are small in comparison to the changes in the vector to the nearby pole. Some applications of this idea are discussed in the problems.

4.3 Potential Analogies

The vectorial interpretation of the pole-zero factors of $H(s)$ helps to make it clear how $|H(s)|$ becomes large for s near a pole and small for s near a zero. Sometimes it is also helpful to think in terms of a physical model. Imagine a rubber sheet with σ and $j\omega$ axes drawn on it. Arrange "tent poles" to push the sheet up at pole locations and "stakes" or "thumbtacks" to tack it down to the ground at zeros. The resulting surface roughly approximates $|H(s)|$. Such an imaginary experiment is often useful for visualizing the effect on $|H(j\omega)|$ of a proposed arrangement of poles and zeros.

An analogous experiment can actually be set up to yield accurate quantitative results. Consider a two-dimensional conducting surface such as a specially coated paper or a shallow tank filled with an electrolyte. Draw a pair of perpendicular axes, labelled σ and ω, on this surface to represent the complex plane. It is an elementary exercise in field theory to show that the potential at a point (σ, ω) on this surface resulting from a current probe touching the surface at the location (σ_0, ω_0) is proportional to $\ln|s - s_0|$, where $s = \sigma + j\omega$ and $s_0 = \sigma_0 + j\omega_0$. If several inward-directed current probes are simultaneously applied to points $(\sigma_{zi}, \omega_{zi})$, several outward-directed current probes are simultaneously applied to points $(\sigma_{pi}, \omega_{pi})$, and the currents are all equal in magnitude, then the potential at (σ, ω) is

$$\phi(\sigma,\omega) \sim \ln\left|\frac{(s - s_{z1})(s - s_{z2})\cdots(s - s_{zM})}{(s - s_{p1})(s - s_{p2})\cdots(s - s_{pN})}\right|.$$

That is, the potential is proportional to the log magnitude of the system function $H(s)$ having the corresponding zero and pole locations.

Today the digital computer has largely replaced such analog techniques for determining the quantitative relation between pole-zero locations and the magnitude and phase of $H(s)$. Yet the qualitative value of potential analogies remains. Sometimes known potential fields can be used without experiment to suggest network designs. Suppose, for example, we want to design a circuit with a frequency response that is approximately constant in magnitude over the band $|\omega| < 2\pi W$ and decays rapidly to zero outside, so that it approximates an ideal lowpass filter such as we shall study extensively in later chapters. The potential field produced by a conducting ring with the collector at infinity, as in Figure 4.3–1a, is an appropriate starting point; from potential theory, the potential will be uniform inside the ring and will fall to a low value outside. We can approximate such a potential with uniformly spaced discrete probes as in Figure 4.3–1b; the corresponding poles would not be a very satisfactory design because half of them are in the right half-plane. However, by symmetry (or by cascading the network with poles as in Figure 4.3–1b with the all-pass structure of Figure 4.3–1c), we conclude that uniformly spaced poles as in Figure 4.3–1d should provide a stable design whose magnitude along the $j\omega$-axis is the square root of the magnitude obtained in Figure 4.3–1b. (See Problem 4.2 for a discussion of all-pass networks.) More densely spaced poles will, of course, provide a better approximation.

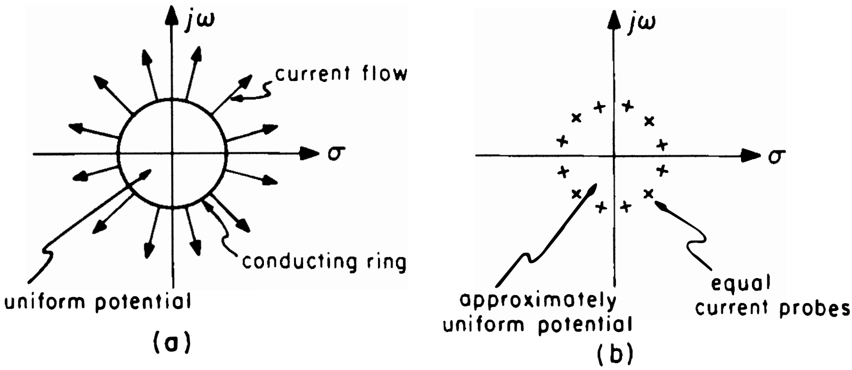

Figure 4.3–1. Pole locations for a lowpass filter.

The resulting filter is one of the class of Butterworth lowpass filters discussed earlier. It can easily be shown to have the frequency response

$$|H(j\omega)|^2 = \frac{1}{1 + \omega^{2n}/(2\pi W)^{2n}}$$

where n is the number of poles. $|H(j\omega)|^2$ is plotted in Figure 4.3–2 for several values of n. (See Problem 4.12 for further discussion.)

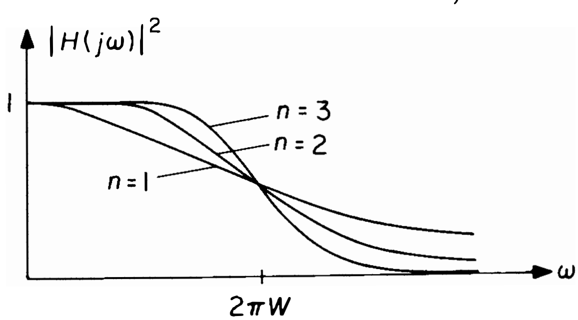

Figure 4.3–2. $|H(j\omega)|^2$ for several Butterworth filters.

4.4 Bode Diagrams

When all of the poles and zeros of $H(s)$ are on the real axis, there exists an important technique—*Bode's method**—for rapidly sketching $|H(j\omega)|$ and $\angle H(j\omega)$ using simple straight-line approximations and asymptotes. Actually a Bode plot is a sketch of $20\log_{10}|H(j\omega)|$ (called the magnitude of $H(j\omega)$ in decibels† (dB)) and $\angle H(j\omega)$ versus $\log_{10}\omega$. A logarithmic amplitude scale not only is useful for displaying phenomena, such as frequency responses, that vary over a wide range, but also may have (at least for analog communication systems) a certain psychological justification, since such sensations as brightness and loudness grow (very roughly) as the logarithm of the intensity. It may be useful to remember that

$$\begin{aligned}1\text{ dB} &= \text{change in } |H(j\omega)| \text{ of about } 12\%,\\ 6\text{ dB} &= \text{change in } |H(j\omega)| \text{ of almost exactly a factor of } 2,\\ 20\text{ dB} &= \text{change in } |H(j\omega)| \text{ of exactly a factor of } 10.\end{aligned}$$

Of course, the reference (0 dB) level need not be selected as $|H(j\omega)| = 1$. More commonly, what is plotted is $20\log_{10}(|H(j\omega)|/H_0)$, where H_0 is some suitable reference level, such as $H(0)$, the maximum value of $|H(j\omega)|$, or some other interesting value. For example, it is customary to measure sound pressures as so many dB above $2\times 10^{-5}\,\text{newton/m}^2$, which is approximately the pressure level corresponding to the average normal threshold of hearing at 1000 Hz.

Bode's scheme is based on the observation that if the poles and zeros are on the σ-axis, we can write

$$H(s) = K\frac{(s+a)(s+b)\cdots}{(s+c)(s+d)\cdots}$$

where K, a, b, c, d, ... are real constants. Assuming for the moment that there are no poles or zeros at $s = 0$, it is convenient to rewrite this in the form

$$H(s) = H(0)\frac{(1+\tau_1 s)(1+\tau_2 s)\cdots}{(1+\tau_3 s)(1+\tau_4 s)\cdots}$$

where τ_i is the *time constant* of the corresponding pole or zero. Taking logarithms yields

$$\begin{aligned}20\log_{10}|H(j\omega)| = 20\log_{10}H(0) &+ 20\log_{10}|1+j\omega\tau_1| + 20\log_{10}|1+j\omega\tau_2| + \cdots\\ &- 20\log_{10}|1+j\omega\tau_3| - 20\log_{10}|1+j\omega\tau_4| - \cdots\end{aligned}$$

and

$$\begin{aligned}\angle H(j\omega) = \angle(1+j\omega\tau_1) &+ \angle(1+j\omega\tau_2) + \cdots\\ &- \angle(1+j\omega\tau_3) - \angle(1+j\omega\tau_4) - \cdots\end{aligned}$$

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*H. W. Bode, *Network Analysis and Feedback Amplifier Design* (New York, NY: Van Nostrand, 1945) Chapter 15.

†The bel (named in honor of the inventor of the telephone) was originally introduced as the logarithm of a power ratio; the decibel (dB) is one-tenth as large. Since power is proportional for most applications to $|H(j\omega)|^2$, we have $10\log_{10}|H(j\omega)|^2 = 20\log_{10}|H(j\omega)|$.

so that if we know the magnitude and angle behavior of a single term $(1+j\omega\tau)$, we can construct magnitude and angle plots by simple addition for any $H(j\omega)$ having poles and zeros on the σ-axis.

The magnitude and angle of $1+j\omega\tau$ are sketched in Figure 4.4–1. The following observations will help in remembering and reproducing such plots:

1. The magnitude plot (dB vs. $\log_{10}\omega$) is asymptotic to two straight lines—a horizontal one at 0 dB for low frequencies and a line with a slope of 6 dB per *octave** (factor of 2) or 20 dB per *decade* (factor of 10) at high frequencies. The two asymptotes intersect at $\omega = 1/\tau$ (called the *breakpoint*).
2. The exact curve differs from the asymptotes by about 3 dB ($\approx 20\log_{10}\sqrt{2}$) at $\omega = 1/\tau$, and by about 1 dB ($\approx 20\log_{10}\sqrt{5/4}$) one octave either side of the breakpoint ($\omega = 2/\tau$ and $\omega = 1/2\tau$).
3. The angle curve goes smoothly from 0° at $\omega = 0$ to 90° at $\omega = \infty$, being 45° at $\omega = 1/\tau$ and being well approximated in between (within about 6°) by a straight line over two decades as shown.

The special cases of poles or zeros at $s = 0$ are handled by simple extensions, as the following example shows.

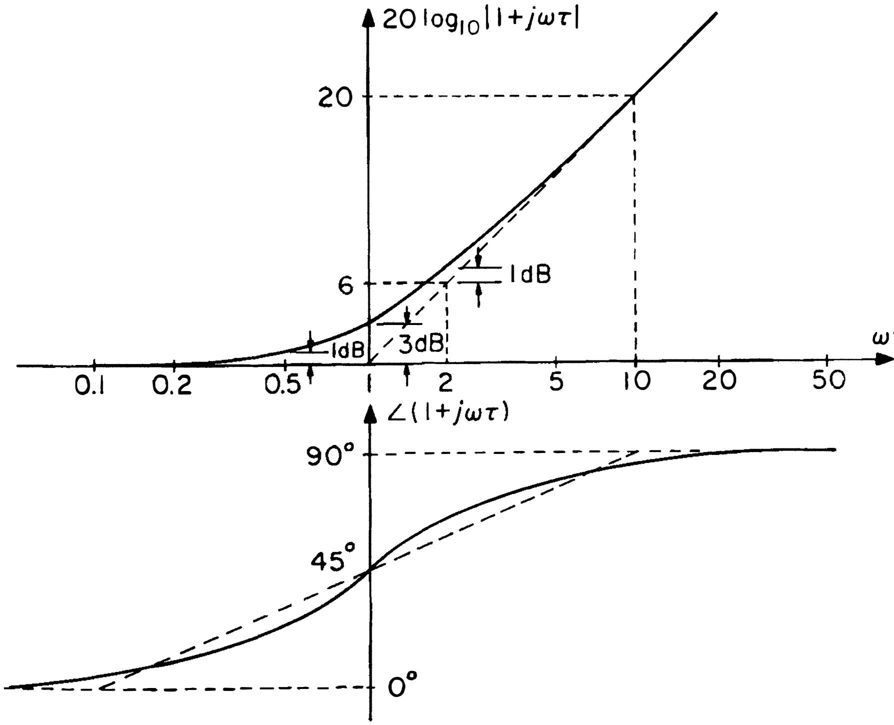


Figure 4.4–1. Magnitude and angle Bode plots for $1+j\omega\tau$.

*The use of the word "octave" for a factor of 2 in frequency derives, of course, from the corresponding musical interval—eight notes on the Western diatonic scale.

Example 4.4–1

To illustrate Bode's method for sketching $|H(j\omega)|$ and $\angle H(j\omega)$, suppose that

$$H(s) = 10\frac{s}{(1+s)\left(1+\frac{s}{10}\right)}.$$

Then

$$20\log_{10}|H(j\omega)| = 20\log_{10}(10) + 20\log_{10}|\omega| - 20\log_{10}|1+j\omega| - 20\log_{10}\left|1+\frac{j\omega}{10}\right|.$$

The asymptotes for each term are shown in the upper part of Figure 4.4–2. The dotted line in the lower part of Figure 4.4–2 shows the superimposed asymptotes; the solid line is a close approximation to the actual response curve for this bandpass filter. The corresponding phase asymptotes and approximate characteristic are shown in Figure 4.4–3.

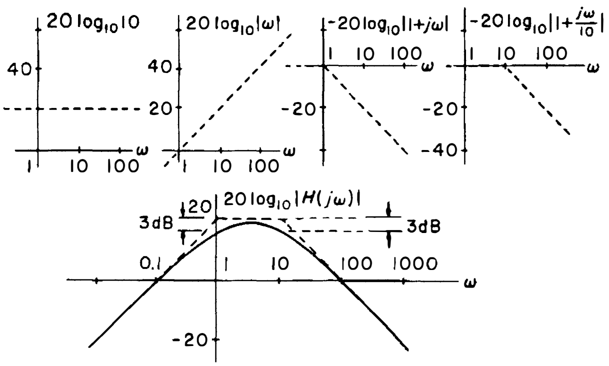


Figure 4.4–2. Bode magnitude plot for Example 4.4–1.

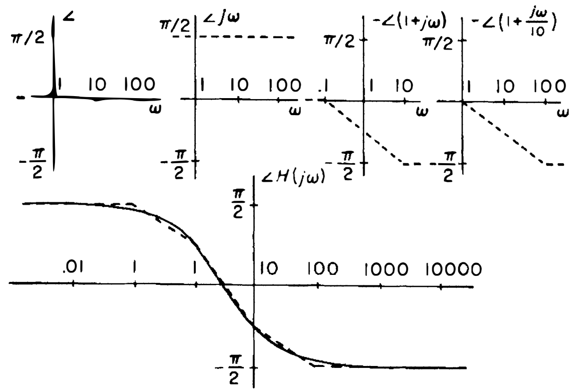


Figure 4.4–3. Bode phase plot for Example 4.4–1.

▶▶▶

Note that every rational function is proportional to some integer power of frequency, $(j\omega)^n$, as ω approaches either 0 or ∞. Hence Bode magnitude plots for large or small frequency are asymptotically straight lines with slopes $6n$ dB/octave (or $20n$ dB/decade). Simultaneously, the angle plots approach $90n$ degrees (modulo 360°). Conversely, if the dB-vs.-$\log_{10}\omega$ plot of an experimentally determined $|H(j\omega)|$ falls at, say, 18 dB/octave as $\omega \to \infty$, then $|H(j\omega)| \sim 1/\omega^3$ at high frequencies. These asymptotic properties apply, of course, to Bode plots for any rational system function, $H(s)$, although the simple techniques we have been describing for constructing such plots are restricted to $H(s)$ having all poles and zeros on the σ-axis.

Example 4.4–2

As a second example, consider

$$H(s) = \frac{\mu\left(1+\dfrac{s}{30}\right)^2}{(1+s)^3}.$$

The Bode diagram for $(1+\tau s)^n$ is simply that for $(1+\tau s)$ multiplied by n. The overall result is shown in Figure 4.4–4.

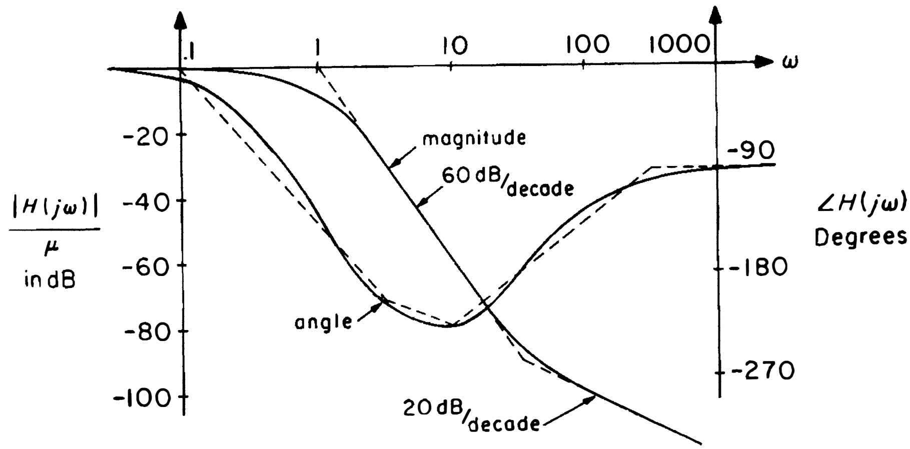


Figure 4.4–4. Bode plots for Example 4.4–2.

▶▶▶

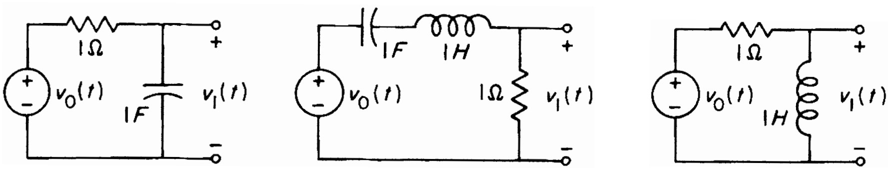
4.5 Summary

A rational system function is completely characterized (except for a real multiplicative constant) by the locations of its poles and zeros. The types of elements that compose the network, as well as the topology of their interconnections, place constraints on what pole-zero locations are possible. Interpreting the pole-zero factors of $H(s)$ as vectors in the complex plane provides insight into the relationship between pole-zero locations and the magnitude and phase of the system function. Special techniques are available for describing the frequency response when one of the poles is close to the $j\omega$-axis (resonance) and when all the poles and zeros are on the real axis (Bode's method).

EXERCISES FOR CHAPTER 4

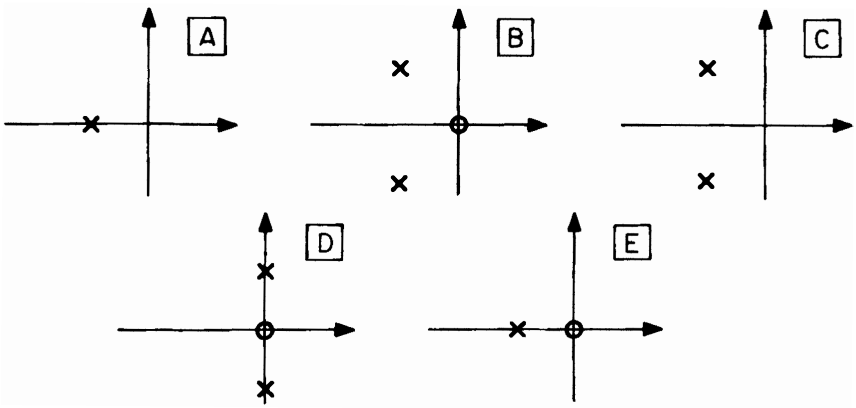
Exercise 4.1

For each of the three circuits below, match a pole-zero plot from (a), an input-output differential equation from (b), a frequency response from (c), and a step response from (d). The correct answers are among those given, and the same answer does not apply to more than one circuit.



(1) (2) (3)

a) Pole-zero plots of the input-output system function:



b) Input-output differential equations:

F $$\frac{d^2v_1(t)}{dt^2}+\frac{dv_1(t)}{dt}+v_1(t)=\frac{dv_0(t)}{dt}$$

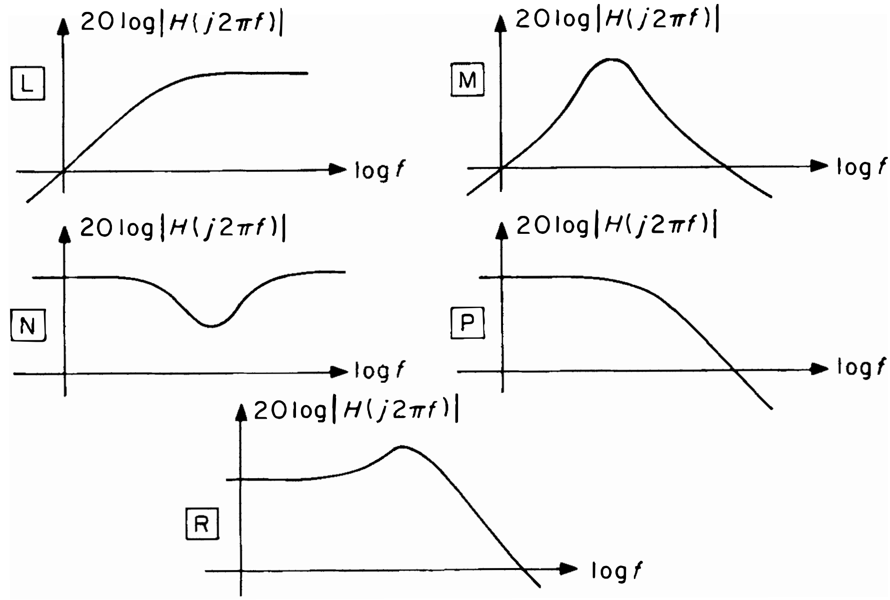
G $$\frac{dv_1(t)}{dt}+v_1(t)=\frac{dv_0(t)}{dt}$$

H $$\frac{d^2v_1(t)}{dt^2}+v_1(t)=\frac{dv_0(t)}{dt}$$

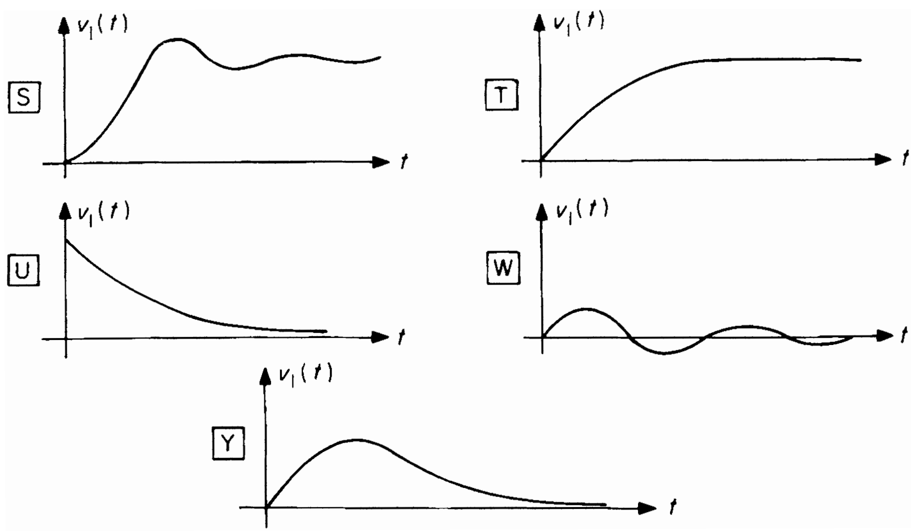
J $$\frac{dv_1(t)}{dt}+v_1(t)=v_0(t)$$

K $$\frac{d^2v_1(t)}{dt^2}+\frac{dv_1(t)}{dt}+v_1(t)=\frac{dv_0(t)}{dt}+v_0(t)$$

c) Input-output frequency responses, $|H(j2\pi f)| = \dfrac{V_1(j2\pi f)}{V_0(j2\pi f)}$:

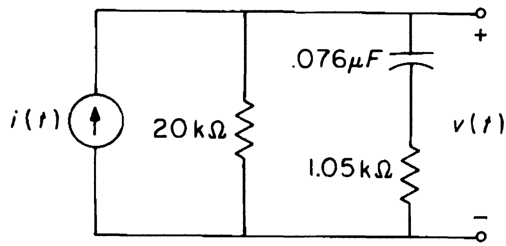


d) Responses to the unit step, $v_0(t) = u(t)$:



The correct answers are: (1) $\boxed{A}\boxed{J}\boxed{P}\boxed{T}$; (2) $\boxed{B}\boxed{F}\boxed{M}\boxed{W}$; (3) $\boxed{E}\boxed{G}\boxed{L}\boxed{U}$.

Exercise 4.2

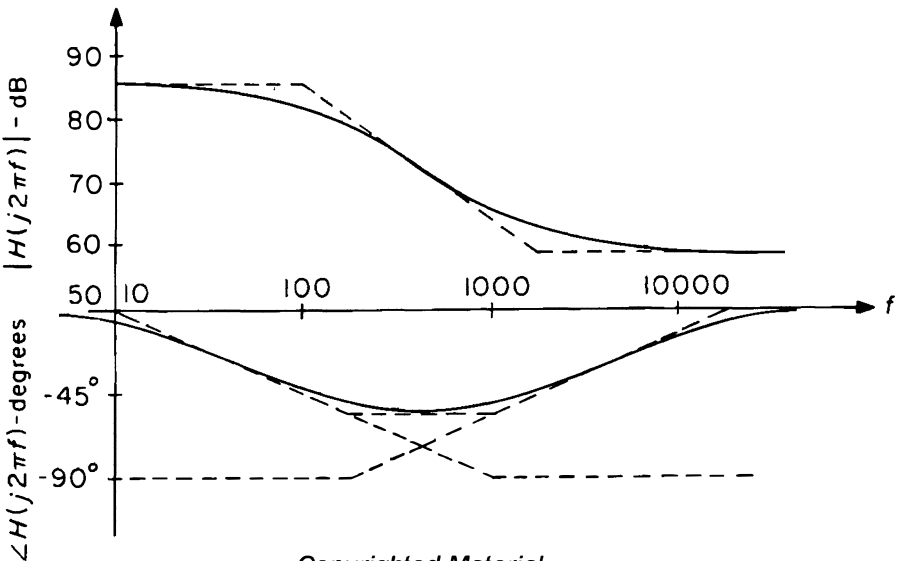


a) Show that the system function of the circuit above is approximately

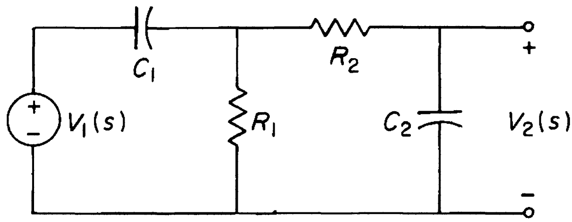
$$H(s) = \frac{V(s)}{I(s)} = 10^3 \frac{s + 2\pi \times 2000}{s + 2\pi \times 100} \qquad \text{(ZSR)}.$$

b) Locate the poles and zeros of $H(s)$ on a sketch of the s-plane.

c) Use Bode's method to show that the magnitude and phase of the frequency response $H(j2\pi f)$ of the above circuit vs. $\log_{10} f$ (where $f = \omega/2\pi$) are as sketched below.



Exercise 4.3



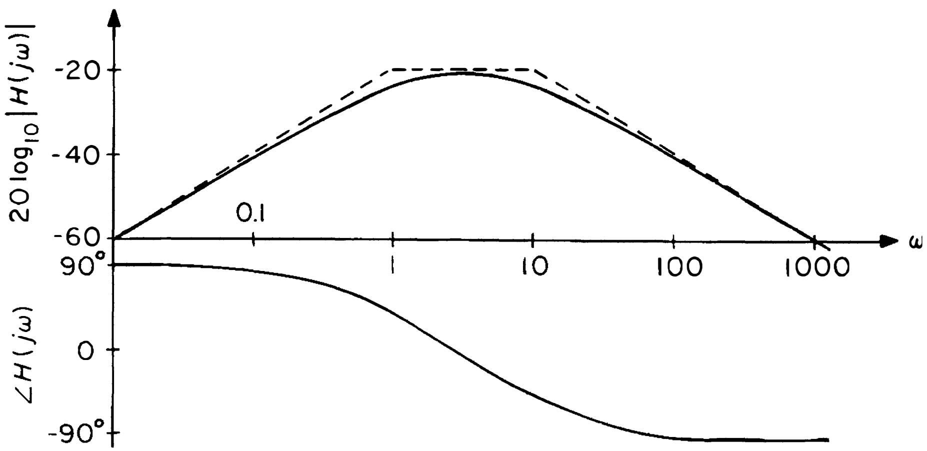
$$R_1 = 1\ \text{k}\Omega, \quad R_2 = 1\ \text{M}\Omega, \quad C_1 = 100\ \mu\text{F}, \quad C_2 = 1\ \mu\text{F}$$

a) From the types of elements and the structure of the network above, argue that both the poles and the zeros of the transfer system function $H(s) = V_2(s)/V_1(s)$ (ZSR) must lie on the σ-axis, although they do not need to alternate.

b) Show that in fact

$$H(s) = \frac{\dfrac{s}{R_2C_2}}{s^2 + \left(\dfrac{1}{R_2C_2} + \dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_1}\right)s + \dfrac{1}{R_1C_1R_2C_2}} \approx \frac{s}{(s+1)(s+10)}.$$

c) Use Bode's method to sketch the magnitude and phase of $H(j\omega)$, checking the curves below. Use limiting arguments at very large and very small frequencies to argue directly from the circuit why a bandpass response of this sort should be expected.



Exercise 4.4

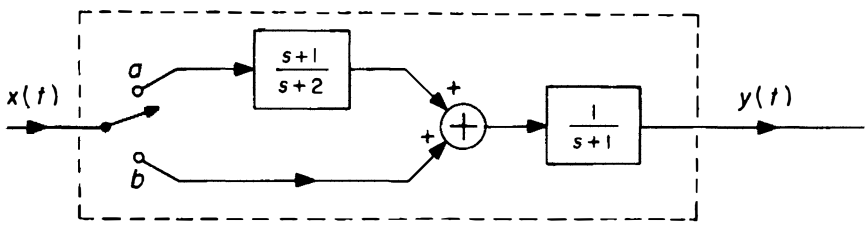
The Laplace transform of the response of a system $H(s)$ to a unit step input is

$$Y(s) = \frac{s+1}{s^3 + 2s^2 + 101s}.$$

Use a pole-zero diagram to argue that the frequency response $|H(j\omega)|$ of this system is similar to that of the simple resonant system of Example 4.2–1 with $Q = 5$.

Exercise 4.5

A system is said to be *controllable* if for every possible initial state there exists an input that will drive the system to rest in a finite time. As an example of an LTI system that is not controllable, consider the system described by the block diagram below.



Prior to $t = 0$, the switch has been in position (b), so that at $t = 0$ the right-hand system will typically have nonzero stored energy, whereas the left-hand system will be at rest. At $t = 0$, the switch is thrown to (a). Using the fact that the response of an LTI system for $t \geq 0$ can be considered as the sum of two parts—ZIR plus ZSR—argue that there is no input $x(t)$ of finite duration (i.e., $x(t) \equiv 0$ for all $t > T$) such that $y(t) \equiv 0$ for all $t > T$.

PROBLEMS FOR CHAPTER 4

Problem 4.1

The system function $H(s) = V_3(s)/V_0(s)$ for the circuit of Example 3.3–1 and Problem 3.1 was found to be

$$H(s) = \frac{\dfrac{-\alpha s}{(1+\alpha)R_1C_1}}{s^2 + \dfrac{s}{1+\alpha}\left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_2} + \dfrac{1}{R_2C_1}\right) + \dfrac{1}{(1+\alpha)R_1C_1R_2C_2}}.$$

a) Show that non-negative values of R_1, R_2, C_1, C_2, and α can always be chosen to position the poles of $H(s)$ at any desired locations on the negative real axis or as a conjugate pair anywhere in the left half-s-plane.

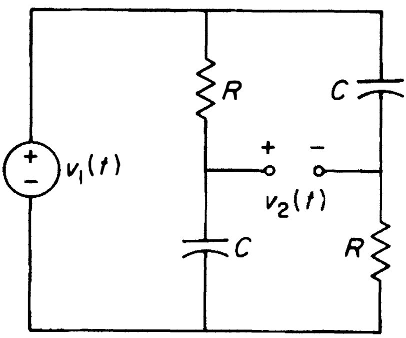
b) In particular show that poles at $s = -10^3$ and -2×10^3 sec^{-1} can be realized with $R_1 = 1000\ \Omega$, $R_2 = 10{,}000\ \Omega$, $\alpha = 1$, and appropriate values of C_1 and C_2.

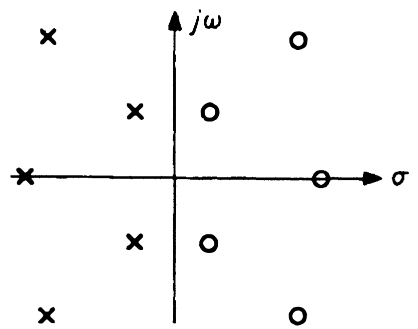
c) Use Bode's method to sketch $|H(j\omega)|$ for the pole locations achieved in (b).

d) Assuming R_1 and R_2 as in (b), find the minimum value of α and appropriate values of C_1 and C_2 to position the poles at $s = -10^2 \pm j10^3$ sec^{-1}.

e) Sketch $|H(j\omega)|$ for the pole locations achieved in (d).

Problem 4.2





a) The circuit above is an example of an *all-pass network*. Compute and sketch $|H(j\omega)|$. Why is this called an "all-pass" network?

b) Using the vectorial interpretation of $H(s)$ developed in Section 4.2, argue that any system function having poles and zeros at mirror image points as shown has the following properties:
   i) $|H(j\omega)| = $ constant
   ii) $\angle H(j\omega)$ is a non-increasing function of ω.

Such a pole-zero diagram is a general characterization of an all-pass network.

Problem 4.3

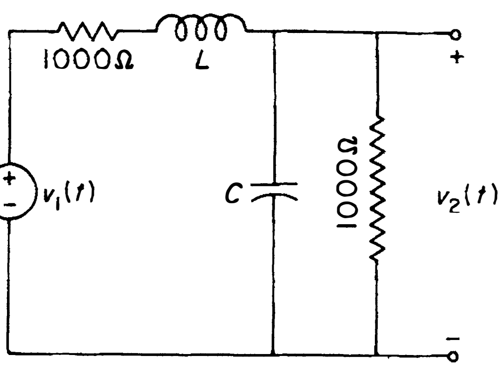
A second-order Butterworth lowpass filter with a cutoff frequency of 6000 rad/sec will have an $H(s)$ described by the pole-zero plot shown below.

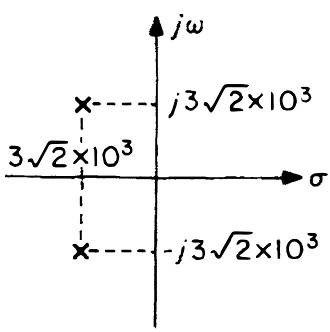
a) Determine analytically and sketch $|H(j2\pi f)|/H(0)$ vs. f. (HINT: It is easier to start by finding $|H(j\omega)|^2$.)

b) Find values of L and C to realize

$$\frac{V_2(s)}{V_1(s)} \sim H(s) \quad \text{(ZSR)}$$

with the form of circuit shown below.





Problem 4.4

Nonzero branch currents and voltages of the form

$$i_\ell = I_\ell e^{s_p t}, \qquad v_\ell = V_\ell e^{s_p t}$$

where s_p is a natural frequency can exist in a circuit under conditions in which all external sources are zero—this, indeed, is the characteristic significance of a natural frequency. This problem derives limitations on the complex numbers s_p that can be natural frequencies for linear circuits composed entirely of positive R's, L's, and C's.

a) Use Tellegen's Theorem* to show that if all the branch voltages and currents have the form above and if all sources and mutual inductances† are zero, then

$$s_p \sum_{\substack{all\\inductors}} L_i |I_i|^2 + s_p^* \sum_{\substack{all\\capacitors}} C_j |V_j|^2 + \sum_{\substack{all\\resistors}} R_k |I_k|^2 = 0.$$

*Tellegen's Theorem states that if $\{v_m\}$ is a complete set of branch voltages in a network that satisfy KVL and $\{i_m^*\}$ is a complete set of branch currents in the same network that satisfy KCL ($\{v_m\}$ and $\{i_m^*\}$ do not have to be sets that would simultaneously satisfy the dynamic equations for the network), then $\sum v_m i_m^* = 0$, where the sum is taken over all branches of the network. (See, e.g., C. A. Desoer and E. S. Kuh, *Basic Circuit Theory* (New York, NY: McGraw-Hill, 1969) p. 393ff.)

†The results to be derived remain true if mutual inductances are not zero, but the proof is slightly more complex.

b) Since each sum above must be real and positive, argue that

i) All natural frequencies of positive RLC networks must have non-positive real parts,

$$\Re e[s_p] \leq 0;$$

ii) The natural frequencies of networks composed of only positive L's and C's *(lossless networks)* must lie on the $j\omega$-axis,

$$\Re e[s_p] = 0;$$

iii) The natural frequencies of networks composed of only positive R's and L's or only positive R's and C's *(relaxation networks)* must all lie on the negative real axis,

$$\Im m[s_p] = 0, \qquad \Re e[s_p] = s_p \leq 0.$$

Problem 4.5

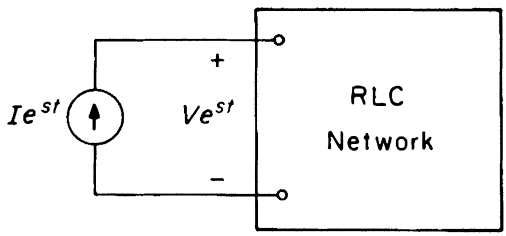
Consider a network driven by a current source of the form Ie^{st}. If $\Re e[s] >$ largest real part of any of the natural frequencies, then ultimately all branch voltages and currents will have the form

$$i_\ell = I_\ell e^{st}, \qquad v_\ell = V_\ell e^{st}.$$

(See Chapter 3.) Furthermore the voltage at the terminals of the network will be

$$Ve^{st} = IZ(s)e^{st}$$

where $Z(s)$ is the driving-point impedance of the network.



a) Use Tellegen's Theorem* in the form $\sum v_\ell i_\ell^* = 0$ (where the sum is to be taken over all branches of the network) to show that if all branch voltages and currents have the form above and if the network is composed of positive R's, L's, and C's (assume that mutual inductances are zero as in the preceding problem), then

$$s\sum_{\substack{all\\inductors}} L_i|I_i|^2 + s^*\sum_{\substack{all\\capacitors}} C_j|V_j|^2 + \sum_{\substack{all\\resistors}} R_k|I_k|^2 = Z(s)|I|^2.$$

*See footnote to Problem 4.4

b) Conclude from the above that for any driving-point impedance,

$$\Re e[Z(s)] \geq 0 \quad \text{for} \quad \Re e[s] \geq 0.$$

(Such a function is called a *positive-real function*.)

c) Interpret this positive-real condition by showing that at any frequency in the sinusoidal steady state a network composed of positive R's, L's, and C's must absorb average power. This is another way of defining a network as passive (see the footnotes in Section 4.1).

Problem 4.6

From the preceding problem, the driving-point impedance of a circuit composed of positive R's and C's only must have the form

$$s^* \sum_{\substack{all \\ capacitors}} C_j |V_j|^2 + \sum_{\substack{all \\ resistors}} R_k |I_k|^2 = Z(s)|I|^2.$$

By analytic continuation this result must hold throughout the entire s-plane, not just to the right of the rightmost natural frequency.

a) Conclude that the driving-point impedance of an RC circuit must satisfy the condition

$$\Im m[Z(s)] \leq 0$$

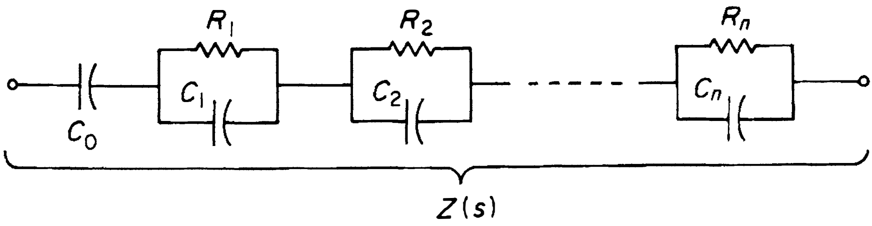
throughout the entire upper half-s-plane.

b) Expand $Z(s)$ in partial fractions, assuming that all poles are simple (it can be shown that this is necessary) in the form

$$Z(s) = \frac{k_0}{s} + \frac{k_1}{s+\sigma_1} + \frac{k_2}{s+\sigma_2} + \cdots + \frac{k_n}{s+\sigma_n}$$

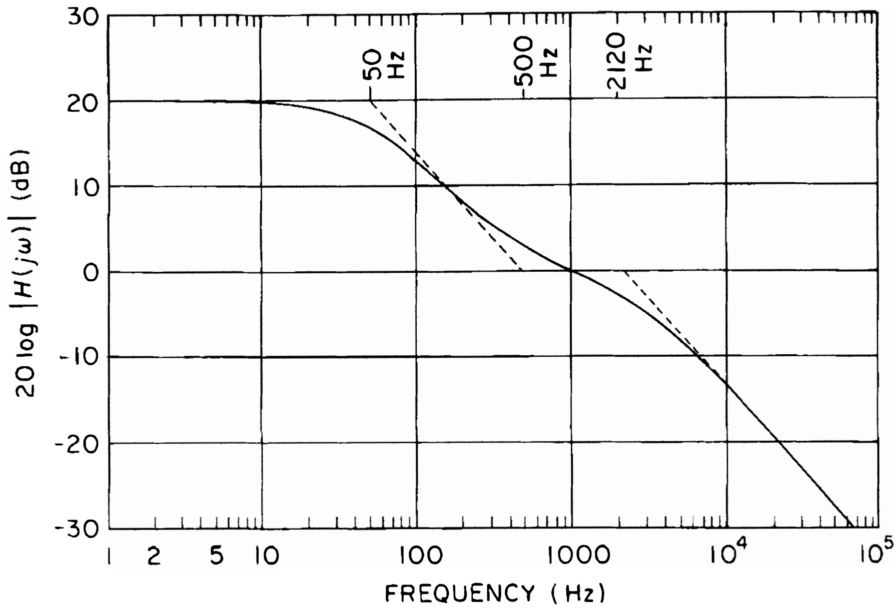
where as previously shown all k_i and σ_i are real and all σ_i are positive. Show that the condition in (a) requires all k_i to be positive. (HINT: If some k_i were negative, consider $\Im m[Z(s)]$ for s in the upper half-plane near $s = -\sigma_i$.)

c) Show that any driving-point impedance having the form given in (b) can be realized in the following circuit arrangement (called the *Foster form*):



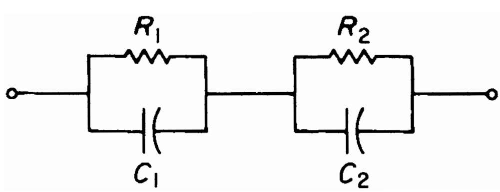
Problem 4.7

Prior to being fed to the recording head that cuts the master for a conventional phonograph record, the audio signal is passed through a network that attenuates the lower frequencies (to avoid overdriving) and boosts the higher frequencies (to help combat scratch noise). A compensating network must then be inserted into the reproducing equipment. The standard RIAA equalizer characteristic is shown in the figure below.

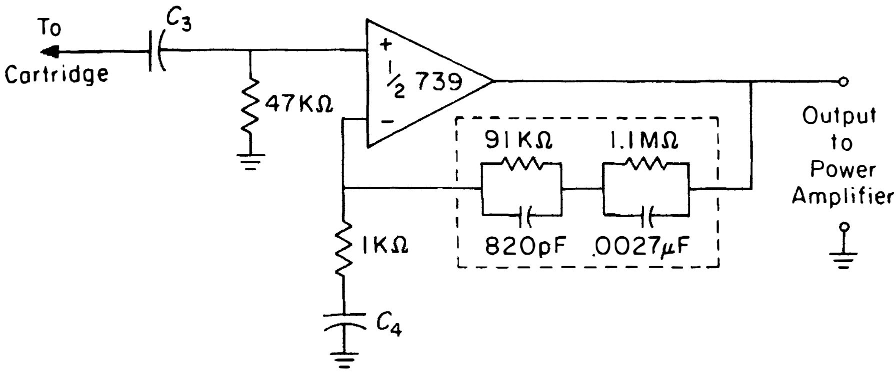


a) Find a simple pole-zero diagram and a rational function $H(s)$ that corresponds to the magnitude characteristic shown. Assume that $H(s)$ contains one zero and two poles, all on the negative real axis.

b) Show that the impedance of the circuit below can be made proportional to $H(s)$ above. Choose the proportionality factor so that the largest resistor is 1.1 MΩ and find all four parameter values.



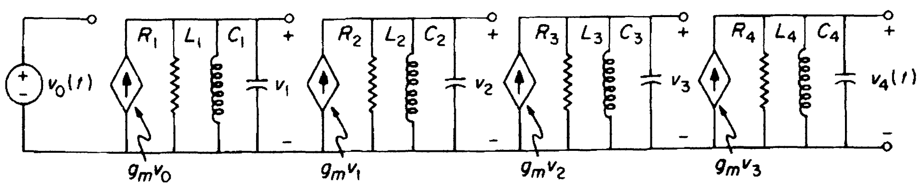
c) A common phonograph-cartridge preamplifier circuit is shown in the figure below. The 739 is a dual low-noise op-amp designed for audio applications. The 47 kΩ resistor is the standard load resistance for magnetic cartridges. C_3 and C_4 are blocking capacitors that are necessary because the 739 is usually powered from a single-sided power supply; assume (for the moment) that they are so large that their effects can be ignored at most frequencies of interest. Show that this circuit approximates the RIAA characteristic. (Note that, except for rounding off to standard 5% values, the impedance of the dashed box is that derived in (b).) What is the gain of this circuit at 1 kHz?



d) In 1978 the RIAA characteristic was modified to specify equalization 3 dB down from the previous standard at 20 Hz, rolling off at 6 dB per octave below 20 Hz. Show that this can be approximately achieved with the above circuit by setting $C_4 = 8.2\ \mu$F. (HINT: Treat the 820 pF and 0.0027 μF capacitors as open circuits at the lower frequencies where C_4 is important, whereas C_4 is approximately a short circuit at the higher frequencies where the other capacitors are important.)

Problem 4.8

The intermediate-frequency (IF) amplifier of a superheterodyne radio receiver has two important functions—to provide the bulk of the amplification needed for the desired station or frequency channel, and to attenuate all other frequency channels. To accomplish these goals, several stages of tuned amplification are usually employed. A simple model for such an amplifier might appear as shown in the figure below.



a) Suppose that all four stages are identical. Argue that

$$|H(j2\pi f)| = \left|\frac{V_4(j2\pi f)}{V_0(j2\pi f)}\right| \approx \frac{K}{[1+4Q^2(1-f/f_0)^2]^2}$$

for f near $f_0 = 1/2\pi\sqrt{LC}$ and for $Q = \omega_0 RC \gg 1$. Find K. (HINT: See Example 4.2–1.)

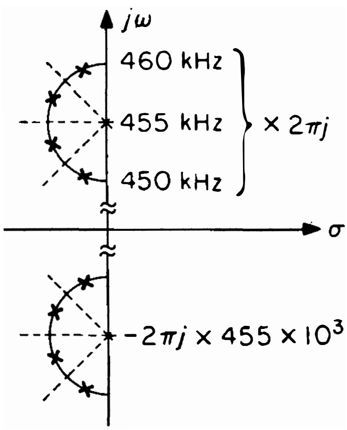
b) Show that the half-power bandwidth of the system of (a) is given by

$$\Delta f = 0.435\frac{f_0}{Q}.$$

Determine the appropriate value of Q for a bandwidth of 10 kHz and a center frequency of 455 kHz. (These are a typical bandwidth and IF frequency for an AM radio receiving station in the standard broadcast band.)

c) Stations other than the one to which the radio is tuned can be represented approximately as sinusoidal components of $v_0(t)$ at frequencies that are multiples of 10 kHz displaced from 455 kHz. Immediately adjacent channels are not usually assigned in the same geographical area. Determine the attenuation in dB of a signal at 475 kHz relative to one at 455 kHz in the system of (a) with Q as in (b).

d) It is usually assumed that an attenuation of 50 to 60 dB is necessary to suppress an unwanted signal adequately; the system described above does not meet these specifications for a station 20 kHz away. Better results can be obtained by tuning the interstage circuits in the figure to slightly different frequencies. Specifically, suppose the poles are equally spaced along a circle as in the diagram to the right. Use an extension of the vectorial ideas and the Butterworth filter characteristics of Section 4.3 to show that



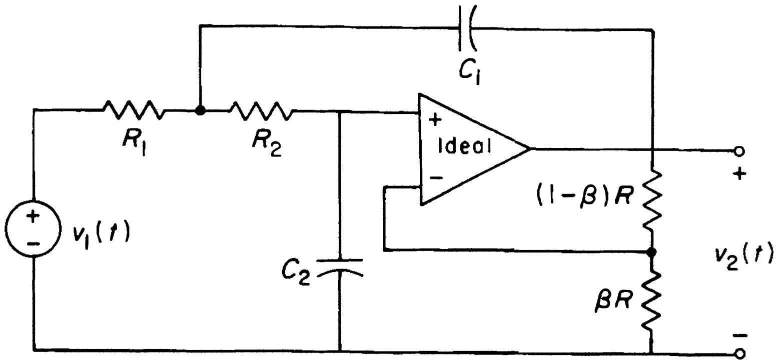
$$\frac{|H(j2\pi f)|}{|H(j2\pi f_0)|} \approx \frac{1}{\sqrt{1+\left[\dfrac{2(f_0-f)}{\Delta f}\right]^8}}$$

where $f_0 = 455$ kHz and $\Delta f = 10$ kHz.

e) The amplifier of (d) has the same half-power bandwidth as that of (a). Find the relative attenuation at 475 kHz for this system. (In practice, coupled coils would be used to realize two pole-pairs in each stage; thus only two stages rather than four would be necessary to achieve this level of performance.)

Problem 4.9

The circuit below is called the *Sallen-Key circuit*.



a) Show that, under ZSR conditions,

$$\frac{V_2(s)}{V_1(s)} = \frac{\omega_0^2/\beta}{s^2 + 2\alpha s + \omega_0^2}$$

where

$$\omega_0^2 = \frac{1}{R_1 C_1 R_2 C_2}$$

$$2\alpha = \frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_2 C_2}\left(1 - \frac{1}{\beta}\right).$$

b) A feature of the Sallen-Key circuit is that the magnitude and angle of the pole locations can be independently adjusted. For the equal-components case ($R_1 = R_2, C_1 = C_2$), show on a sketch of the s-plane the locus of pole locations that can be realized for a fixed value of R_1C_1 as β varies, $0 < \beta < 1$. How does this locus change as the value of R_1C_1 changes?

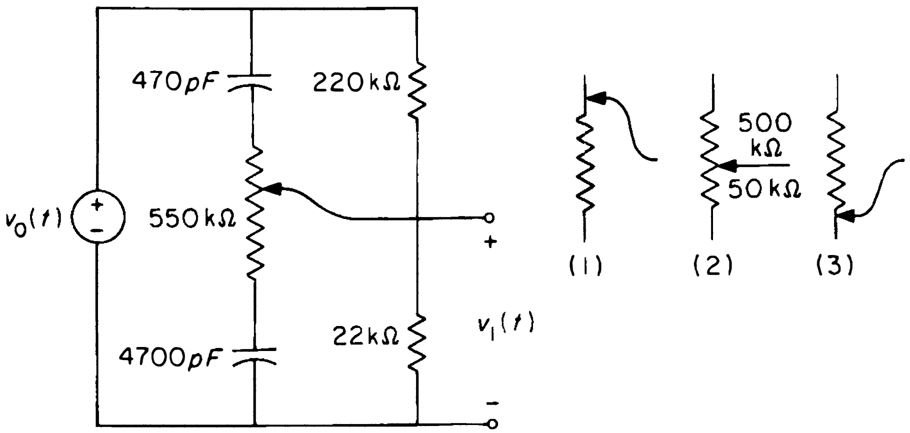
c) The parameter values in a cascade of several Sallen-Key circuits can be chosen to realize an overall system with arbitrary pole locations and all zeros at ∞. Illustrate this procedure by finding the element values for a cascaded pair of Sallen-Key circuits whose overall system function has the form of a 4^{th}-order Butterworth lowpass filter with cutoff frequency of 1 kHz, that is,

$$H(s) = \frac{K}{(s^2 + 11.610 \times 10^3 s + 39.478 \times 10^6)(s^2 + 4.809 \times 10^3 s + 39.478 \times 10^6)}.$$

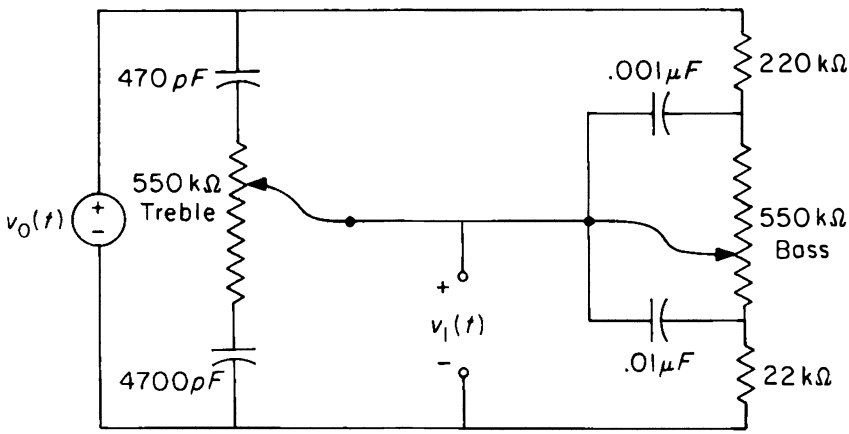
For simplicity, choose all capacitors to be 0.1 μF. What is the resulting value of K? Show the locations of the poles in the complex plane and explain why this pole-zero arrangement would be expected to correspond to a lowpass filter with 1 kHz cutoff frequency. What would be the effect on $|H(j\omega)|$ of a change in the value of all the capacitors to 0.2 μF?

Problem 4.10

a) A typical *treble tone-control circuit*, providing both treble "boost" and "cut," is shown in the figure on the next page. Sketch the magnitude of the frequency response of this circuit for the three positions of the potentiometer shown in (1), (2), and (3). Make reasonable approximations.

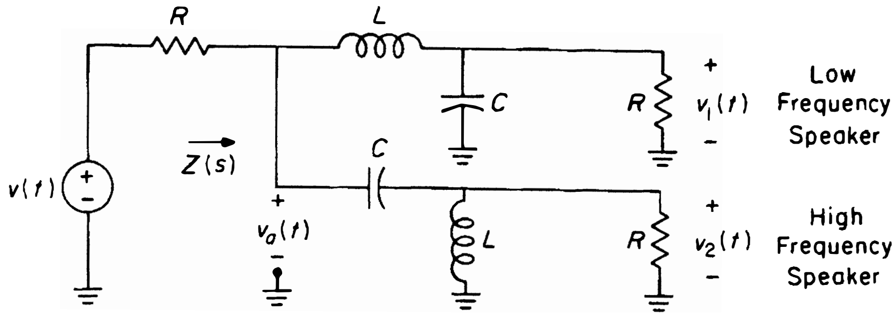


b) Discuss the "taper" of the potentiometer—the curve of resistance between the tap and one end as a function of geometric tap position or shaft angle—that would probably be appropriate for this circuit. (With the addition of several more elements this circuit can provide both treble and bass control as shown below.)



Problem 4.11

The circuit shown on the next page is frequently employed as a *crossover network* to apply the appropriate ranges of frequencies to the low- and high-frequency speakers (represented by the resistances R) of a hi-fi system.



$$L = \frac{\sqrt{2}R}{\omega_0}; \qquad C = \frac{1}{\sqrt{2}R\omega_0}$$

a) Show that the impedance $Z(s)$ looking into the crossover network is in fact a constant resistance R, independent of frequency.

b) Find the system functions relating each of the outputs to the input. Locate their poles and zeros on a sketch of the s-plane. (HINT: Note that as a result of (a), the voltage $v_a(t)$ is simply a fixed fraction of $v_0(t)$, independent of frequency. You should find that the poles of both system functions are located at $s = \omega_0 e^{\pm j3\pi/4}$.)

c) Sketch the magnitudes of the frequency transfer characteristics $|V_i(j\omega)/V(j\omega)|$ of the two speakers. What is the significance of the parameter ω_0?

Problem 4.12

If $H(s)$ is a rational function (i.e., a ratio of polynomials in s with real coefficients), then it is easy to show that $|H(j\omega)|^2$ will be a rational function in ω^2 with real coefficients. To go backwards from $|H(j\omega)|^2$ to $H(s)$ one can use the following procedure:

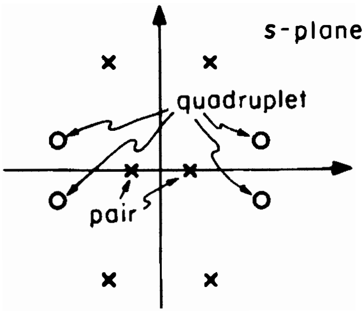
i) Observe that

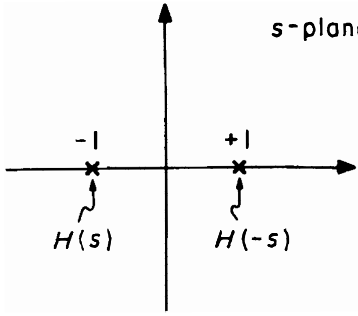
$$|H(j\omega)|^2 = H(j\omega)H(-j\omega) = [H(s)H(-s)]_{s=j\omega}\,.$$

Hence, if we substitute $\omega^2 = -s^2$ in $|H(j\omega)|^2$, we can identify the result as $H(s)H(-s)$.

ii) Locate the poles and zeros of $H(s)H(-s)$ on a sketch of the s-plane. In general they will occur in quadruplets, except that poles or zeros on the real axis need occur only in pairs, as shown on the left in the figure on the next page.

iii) Identify the poles in the left half-plane with $H(s)$. Their images in the right half-plane then correspond to $H(-s)$. The zeros of $H(s)$ may be taken (in conjugate pairs) from either half-plane; if they are all taken in the left half-plane, the result is called the *minimum-phase* $H(s)$.





As an example consider

$$|H(j\omega)|^2 = \frac{1}{1+\omega^2} = \left.\frac{1}{1-s^2}\right|_{s=j\omega}$$

Then

$$H(s)H(-s) = \frac{1}{1-s^2} = \frac{1}{(1+s)(1-s)}$$

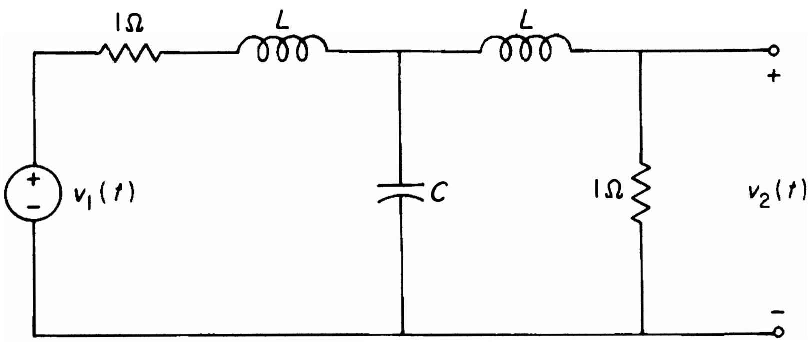
which yields the diagram shown to the right above. Thus

$$H(s) = \frac{1}{s+1}, \qquad H(-s) = \frac{1}{-s+1}.$$

a) Apply this procedure to find $H(s)$ corresponding to the third-order Butterworth lowpass filter (see Example 1.7–2):

$$|H(j\omega)|^2 = \frac{1}{1+\omega^6}.$$

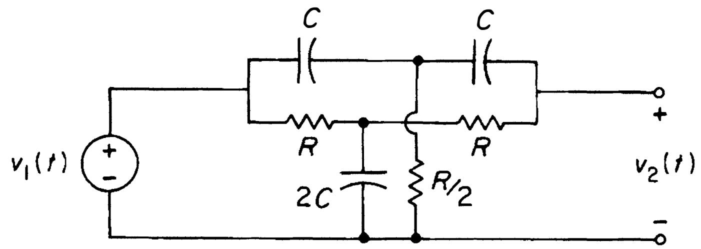
b) Show that this $H(s)$ can be realized (except for a scale factor) by the circuit below, which was studied extensively in earlier chapters. Find the values of L and C.



c) Use the result obtained in (b) to determine values of L and C for a practical filter to work between resistors having the value of 1000 Ω instead of 1 Ω and yielding a cutoff frequency of 1000 Hz instead of 1 rad/sec.

Problem 4.13

The Twin-T network of Example 4.1–1 is to be used as a "notch" filter to eliminate a 60 Hz powerline interference signal from the voltage-source waveform $v_1(t)$.



a) From the results of Example 4.1–1, show that

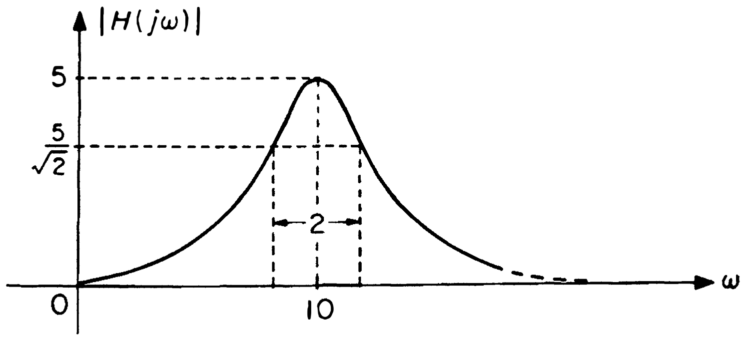
$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{s^2 + \dfrac{1}{(RC)^2}}{s^2 + \dfrac{4s}{RC} + \dfrac{1}{(RC)^2}}.$$

b) Indicate the pole and zero locations for $H(s)$ on a sketch of the s-plane.

c) Show that $|H(j\omega)| = 0.707|H(0)|$ for $\omega/RC = \sqrt{5} \pm 2$. These points, together with the limiting values of $|H(j\omega)|$ as $\omega \to 0$ and $\omega \to \infty$ and the location of the zero, should give you enough information to prepare an adequate sketch of $|H(j\omega)|$ vs. $\log_{10}(\omega/RC)$.

d) Determine an appropriate value of R to place the zero at 60 Hz if $C = 1$ μF.

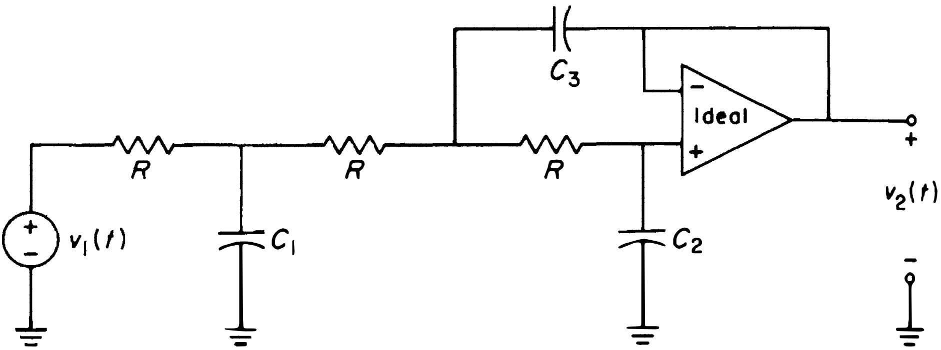
Problem 4.14



a) Find the poles and zeros of a system function $H(s)$ having approximately the frequency response shown on the preceding page.
b) Find a differential equation relating the input $x(t)$ and the output $y(t)$ of the LTI system described in (a).

Problem 4.15

The following circuit appears in an op-amp manufacturer's data manual with the suggestion that it may be useful as a scratch filter in a phonograph amplifier.



a) Show that the system function $H(s) = V_2(s)/V_1(s)$ (ZSR) is given by

$$H(s) = \frac{1}{(RC_1RC_2RC_3)s^3 + 2RC_2(RC_1 + RC_3)s^2 + (RC_1 + 3RC_2)s + 1}.$$

b) Locate the poles of $H(s)$ on a plot of the s-plane for $C_1 = 0.0022\ \mu\text{F}$, $C_2 = 330$ pF, $C_3 = 0.0056\ \mu\text{F}$, $R = 10\ \text{k}\Omega$. (HINT: One of the poles should be near $s = -2\pi \times 10^4$ sec^{-1}.)
c) Sketch $|H(j\omega)|$. (HINT: Consider $H(s)$ as approximating one of the class of Butterworth lowpass filters described in Section 4.3.)
d) The op-amp manufacturer suggests that the half-power frequency of this lowpass filter (the frequency at which $|H(j\omega)|^2 = 0.5$) can be adjusted by using a 3-ganged switch or potentiometer to change the common value, R, of the three resistors. Explain how this works in terms of the algebraic structure of $H(s)$. Describe the pole locations and determine the value of the half-power frequency if R is doubled (to 20 kΩ) with no change in the values of the capacitors. Repeat for R halved (to 5 kΩ). (It should be possible to answer these questions without recomputing or refactoring the denominator polynomial of $H(s)$.)