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6.3100 Lecture 2 Notes – Spring 2023

General solutions to first-order DT system, stability and convergence

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Outline:

1. Proportional control for first order discrete time system
2. Solutions to first order discrete time systems
3. Choosing Kp for a first order system: stability, steady-state error, and convergence

1. Proportional control for first order discrete time system

In the previous lecture, we introduced a simple first order discrete time (DT) system and the proportional controller. As a reminder, the controller and the first order system equations are given by:

Proportional controller: $u[n] = K_p(T_d[n] - T_m[n])$

Plant: $\frac{T_m[n]-T_m[n-1]}{\Delta T} = \gamma u[n-1]$

We can substitute the first equation into the second equation:

$$\frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma K_p(T_d[n-1] - T_m[n-1])$$

Simplifying this equation and collecting terms, we obtain the expression:

$$T_m[n] = (1 - \gamma K_p \Delta T)T_m[n-1] + \gamma \Delta T K_p T_d[n-1]$$

This equation has the form of a 1^{st}-order DT system. We can write the general form as:

$$y[n] = \lambda y[n-1] + bx[n-1] \quad (\#1)$$

Here $y[n]$ is the variable we aim to solve, $x[n]$ is the input (driving) function we set, λ is the natural frequency (we will explain why later), and b is a multiplicative constant. In the next section, we will study the solution and property of equation 1 in detail.

2. Solutions to first order discrete time systems

We are going to solve equation (1) for several cases.

Case 1: x[n]= 0 for all n. This is called zero-input response (ZIR)

The equation simplifies to $y[n] = \lambda y[n-1]$.

The solution of this problem is given by:

$$y[n] = \lambda^n y[0]$$

This is a very simple case. Note that the steady state solution depends on the value of λ.

If $|\lambda| < 1$, then $y[\infty] = 0$.

If $\lambda = 1$, then $y[\infty] = y[0]$.

If $\lambda = -1$, then $y[n] = (-1)^n y[0]$. The solution does not converge.

If $|\lambda| > 1$, then $|y[\infty]| \to \infty$. The solution does not converge.

Case 2: x[n]= 1 for all n, and y[0]=0. This is called zero-state response (ZSR).

Note: x[n] = 1 is not limiting. Through invoking linearity and time invariance (next lecture), we can relax the solution form by letting x[n] be any arbitrary function.

In this case, equation (1) becomes

$$y[n] = \lambda y[n-1] + b$$

First, assuming the solution converges, let us find $y[\infty]$. We have

$$y[\infty] = \lambda y[\infty] + b$$

$$y[\infty] = \frac{b}{1-\lambda}$$

Next, let's find $y[n]$. We can write y[n] iteratively, as:

$$y[0] = 0$$
$$y[1] = \lambda y[0] + b = b$$
$$y[2] = \lambda y[1] + b = \lambda b + b$$
$$y[3] = \lambda y[2] + b = \lambda^2 b + \lambda b + b$$

Following this pattern, we get:

$$y[n] = \sum_{m=0}^{n-1} \lambda^m b \text{ and } y[\infty] = \sum_{m=0}^{\infty} \lambda^m b$$

This implies

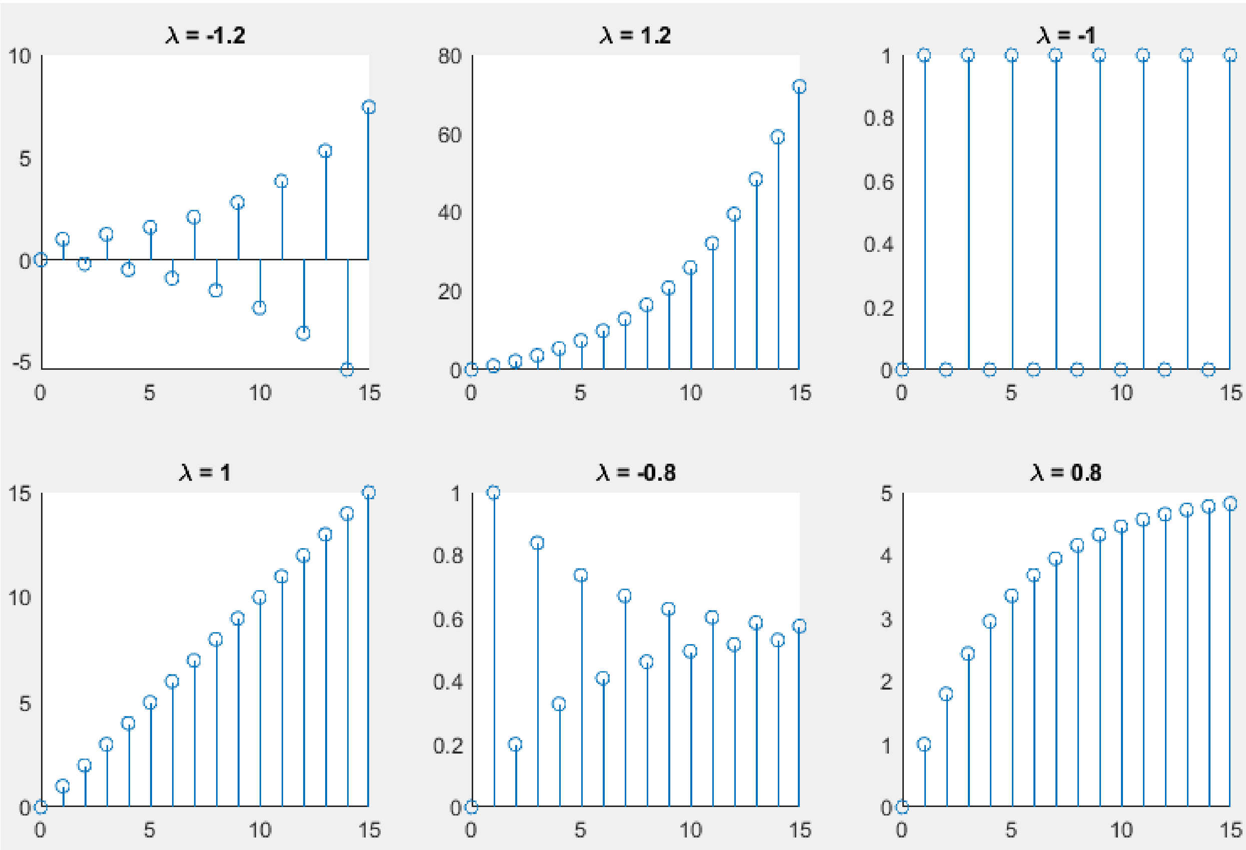
$$y[n] = y[\infty] - \sum_{m=n}^{\infty} \lambda^m b = y[\infty] - \lambda^n \sum_{m=0}^{\infty} \lambda^m b = y[\infty] - \lambda^n y[\infty] = y[\infty](1-\lambda^n)$$

Substituting the solution of $y[\infty]$, we obtain:

$$y[n] = \frac{b}{1-\lambda}(1-\lambda^n)$$

Let's interpret what the solution looks like. Suppose b = 1, we consider 6 scenarios:

(1) $\lambda > 1$. Solution diverges
(2) $\lambda < -1$. Solution diverges
(3) $\lambda = -1$. Solution diverges
(4) $\lambda = 1$. Solution diverges
(5) $0 < \lambda < 1$. Solution converges
(6) $-1 < \lambda < 0$. Solution converges



Now we know how to solve 1st order DT systems, let's return to our 3D-printing controller example. It's worthwhile to emphasize again that the value of λ is crucial for the solution to either diverge or converge. In a control system, we need to design stable systems through setting the value of λ.

3. Choosing Kp for a first order system: stability, steady-state error, and convergence

Returning to the 3D-printing example, the system equation is given by:

$$T_m[n] = (1 - \gamma K_p \Delta T) T_m[n-1] + \gamma \Delta T K_p T_d[n-1]$$

The key question is how should we choose K_p to construct a "good" controller?

First, we can pattern-match to find λ and b. We have:

$$\lambda = 1 - \gamma K_p \Delta T$$
$$b = \gamma \Delta T K_p T_d[n]$$

Here we can assume the desired temperature is constant.

There are several key metrics we need to consider:

(1) Stability:

$$-1 < \lambda < 1$$
$$-1 < 1 - \gamma K_p \Delta T < 1$$
$$\frac{2}{\gamma \Delta T} > K_p > 0$$

For this control problem, Kp must be chosen in the desired range to guarantee system stability (that is $T_m[\infty]$ is a finite number).

(2) Steady-state error:

We can use the steady-state solution to evaluate if there is any steady state error. We have:

$$T_m[\infty] = y[\infty] = \frac{b}{1-\lambda} = \frac{\gamma \Delta T K_p T_d[\infty]}{1-(1-\gamma K_p \Delta T)} = T_d[\infty]$$

In this particular problem, $T_m[\infty] = T_d[\infty]$. As long as the system is stable, then there is no steady-state error. This is only true for this particular example. In the next class, we are going to see an example where Kp influences the steady state error.

(3) Convergence rate:

Thus far, the two conditions only give us a range of valid Kp. What is the optimal Kp? There are many metrics to optimize for. In this example, let's consider the goal of making the measured temperature $T_m[n]$ approach its desired value $T_d[n]$ as soon as possible. Going back to the general solution:

$$y[n] = \frac{b}{1-\lambda}(1-\lambda^n)$$

What if we let $\lambda = 0$? Then we have:

$$y[1] = \frac{b}{1}(1) = b$$

This is a very nice result because the temperature approaches the desired value in 1 step. This is very fast convergence. Realistically, it may be influenced by external noises, and it usually requires a large control input. Those tradeoffs are things we need to consider when designing a realistic controller.