6.3100: Dynamic System Modeling and Control Design

Discrete-Time System Functions

Polynomial (aka Transform) Representations of Systems

February 27, 2023

Representations of Discrete-Time Systems

Different representations of systems facilitate different insights.

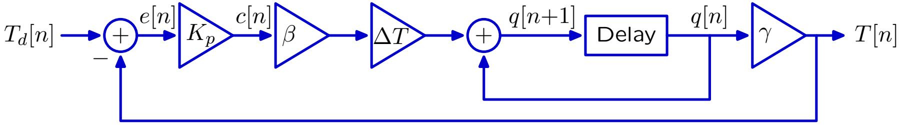
Verbal descriptions can capture the physics of a problem.

"The input to a furnace sets the rate at which it generates heat."

Difference equations are mathematically concise.

$$T[n+1] = (1-\gamma\Delta T\beta K_p)T[n] + \gamma\Delta T\beta K_p T_d[n]$$

Block diagrams illustrate pathways through which signals flow.

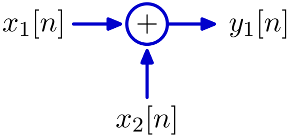


Today we will introduce a new representation in which

- relations among signals are represented by **polynomials**, and
- an entire system is represented by a ratio of polynomials called the **system function**.

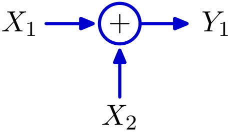
From Samples to Signals

Rather than thinking about relations among samples:



$$y_1[n] = x_1[n] + x_2[n]$$

we will think about relations among signals:



$$Y_1 = X_1 + X_2$$

where X_1 represents an entire signal: $x_1[0]$, $x_1[1]$, $x_1[2]$, ...
X_2 represents an entire signal: $x_2[0]$, $x_2[1]$, $x_2[2]$, ...
Y_1 represents an entire signal: $y_1[0]$, $y_1[1]$, $y_1[2]$, ...

Notice that the addition operators for samples and signals are **different**.

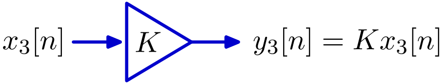
The former (top) adds two samples and generates a new sample.

The latter (bottom) adds two signals and generates a new signal.

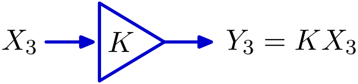
From Samples to Signals

We can similarly define operators to **scale** and **delay** a signal.

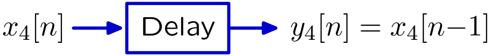
Scaling samples:



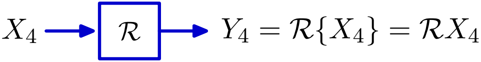
becomes scaling signals:



Delaying samples:



becomes delaying signals:



where the $\mathcal{R}$ **operator** shifts its input signal to the **right** by one sample.

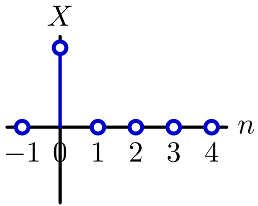
Operator Notation: Check Yourself

Let $Y = \mathcal{R}X$. Which of the following is/are true:

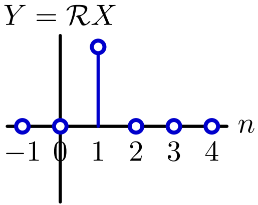
1. $y[n] = x[n]$ for all n
2. $y[n+1] = x[n]$ for all n
3. $y[n] = x[n+1]$ for all n
4. $y[n-1] = x[n]$ for all n
5. none of the above

Check Yourself

Consider a simple signal:



Then



Clearly $y[1] = x[0]$. Equivalently, if $n = 0$, then $y[n+1] = x[n]$.
The same sort of argument works for all other n.

Operator Notation: Check Yourself

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4. $y[n-1] = x[n]$ for all n
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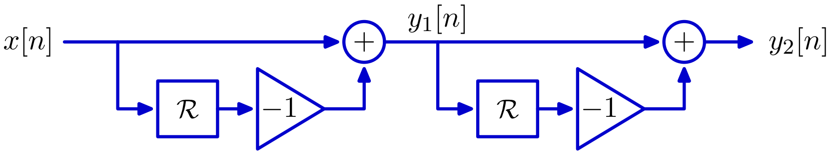
Polynomial (Functional) Representations

Instead of **difference equations** to specify relations among **samples**, we use **polynomials** in $\mathcal{R}$ to specify relations among **signals**.



Start with the difference equations:

$$\begin{aligned} y_2[n] &= y_1[n]-y_1[n-1] \\ &= (x[n]-x[n-1]) - (x[n-1]-x[n-2]) \\ &= x[n] - 2x[n-1] + x[n-2] \end{aligned}$$

The equivalent operator representation has the same structure:

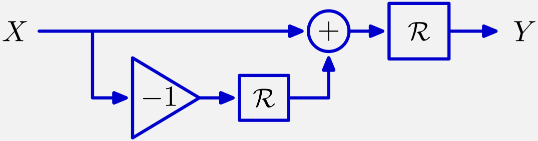
$$\begin{aligned} Y_2 &= (1-\mathcal{R})\{Y_1\} = (1-\mathcal{R})\{(1-\mathcal{R})\{X\}\} = (1-\mathcal{R})(1-\mathcal{R})\,X \\ &= (1-\mathcal{R})^2 X \\ &= (1-2\mathcal{R}+\mathcal{R}^2)\,X \end{aligned}$$

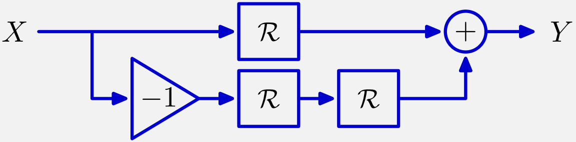
Notice that the polynomial representation retains much of the structure of the difference equations.

Check Yourself

Operator expressions obey many of the algebraic rules of polynomials. The following systems are described by the same difference equation:

$$y[n] = x[n-1] - x[n-2]$$

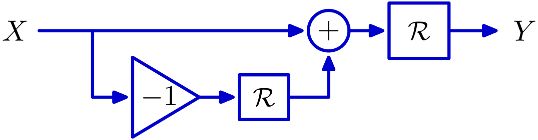




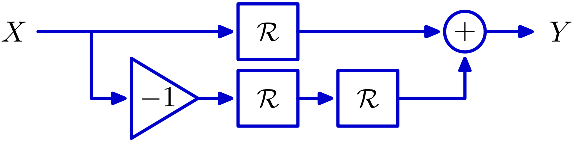
Their operator expressions are related by what math property?

1. commutativity
2. associativity
3. distributivity
4. transitivity
5. none of the above

Check Yourself



$$Y = \mathcal{R}(1 - \mathcal{R})X$$



$$Y = (\mathcal{R} - \mathcal{R}^2)X$$

Multiplication by $\mathcal{R}$ distributes over addition.

Check Yourself

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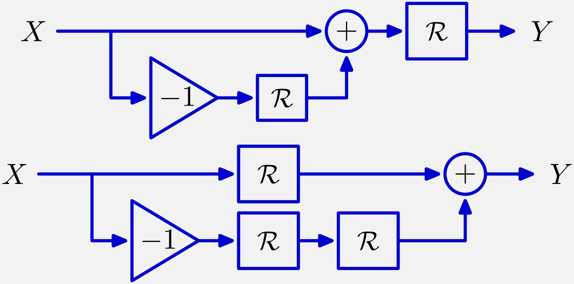
$$y[n] = x[n-1] - x[n-2]$$



Their operator expressions are related by what math property?

1. commutativity
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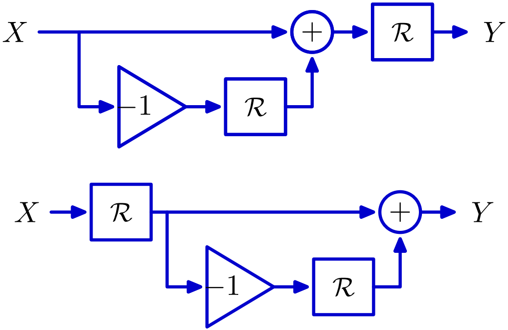
Operator Algebra

Similarly, operator expressions obey the commutativity principle:

$$\mathcal{R}(1-\mathcal{R})\,X = (1-\mathcal{R})\mathcal{R}X$$



These systems are equivalent in the sense that they are described by the same difference equation:

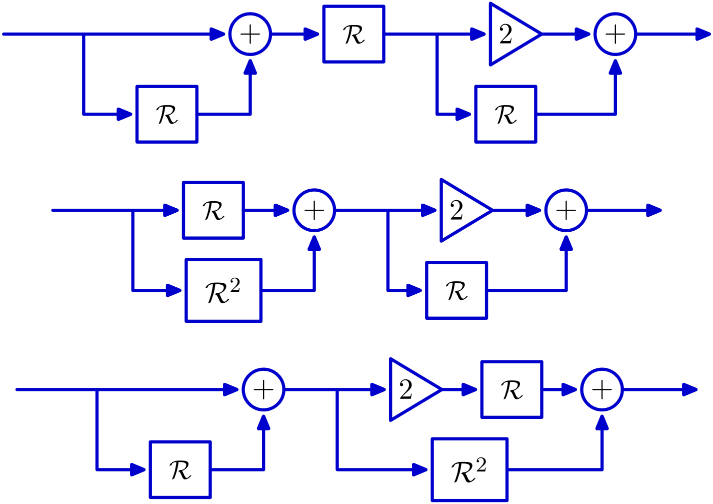
$$y[n] = x[n-1] - x[n-2]$$

Operator Algebra

The associative property similarly holds for operator expressions.

$$(2+\mathcal{R})\mathcal{R}(1+\mathcal{R}) = (2+\mathcal{R})\Big(\mathcal{R}(1+\mathcal{R})\Big) = \Big((2+\mathcal{R})\mathcal{R}\Big)(1+\mathcal{R})$$

Corresponding block diagrams:



Using Operator Representations

Operator expressions obey the usual **rules of algebra** for polynomials.

They are useful for **manipulating** (and simplifying) system representations.

They are also useful for **evaluating** input and output signals.

Check Yourself

Consider the system described by the following operator expression:

$$Y = (1+3\mathcal{R})\mathcal{R}X$$

Determine the output Y when the input X is the following signal:

where $x[n] = 0$ for n outside the range shown above.

For what value of n (if any) is $y[n] = 4$?

0: 0 1: 1 2: 2 3: 3 4: 4

none of the above

Check Yourself

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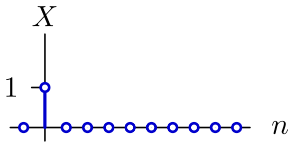
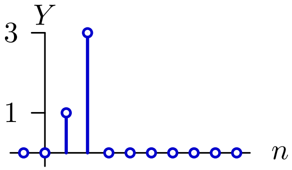


$$Y = (1+3\mathcal{R})\mathcal{R}X = \mathcal{R}X + 3\mathcal{R}^2X$$



Check Yourself

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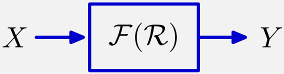
For what value of n (if any) is $y[n] = 4$? 5: none of the above

0: 0 1: 1 2: 2 3: 3 4: 4

5: none of the above

Check Yourself

Consider the system described by its operator representation $\mathcal{F}(\mathcal{R})$:



where $\mathcal{F}(\mathcal{R}) = \mathcal{R}+3\mathcal{R}^2$.

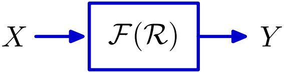
Determine the output Y when X is a geometric sequence

$$x[n] = z^n$$

where z is a (possibly complex-valued) constant.

Check Yourself

Consider the system described by its operator representation $\mathcal{F}(\mathcal{R})$:



where $\mathcal{F}(\mathcal{R}) = \mathcal{R}{+}3\mathcal{R}^2$.

Determine the output Y when the input X is a geometric sequence

$$x[n] = z^n$$

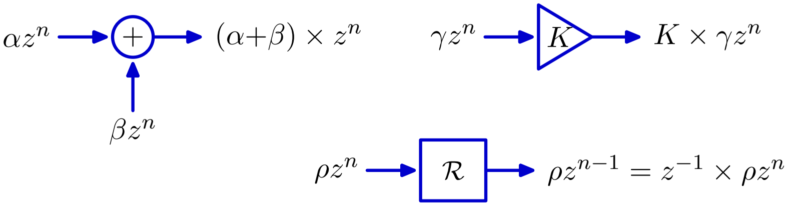
where z is a (possibly complex-valued) constant.

$$\begin{aligned} Y &= \mathcal{F}(\mathcal{R})\{X\} \\ &= \underbrace{(\mathcal{R} + 3\mathcal{R}^2)}_{\mathcal{F}(\mathcal{R})}\{z^n\} = \mathcal{R}\{z^n\} + 3\mathcal{R}^2\{z^n\} = z^{n-1} + 3z^{n-2} \\ &= \underbrace{\left(z^{-1} + 3z^{-2}\right)}_{\mathcal{F}(z^{-1})} z^n = \mathcal{F}(z^{-1})z^n = \mathcal{F}(z^{-1})X \end{aligned}$$

The output signal is geometric with the same base as the input. It is thus a scaled version of the input X, where the scale factor is $\mathcal{F}(\mathcal{R})\Big|_{\mathcal{R}\to\frac{1}{z}}$.

Geometric Signals

When the inputs to adders, gains, and delays are proportional to z^n, their outputs are also proportional to z^n.



Similarly if the input to any combination of adders, gains, and delays is proportional to z^n, then the output is also proportional to z^n.

To find the constant of proportionality, simply substitute $\frac{1}{z}$ for $\mathcal{R}$ in the corresponding operator expression:

$$H(z) = \mathcal{F}(\mathcal{R})\Big|_{\mathcal{R}\to\frac{1}{z}}$$

$H(z)$ is called the **system function**.

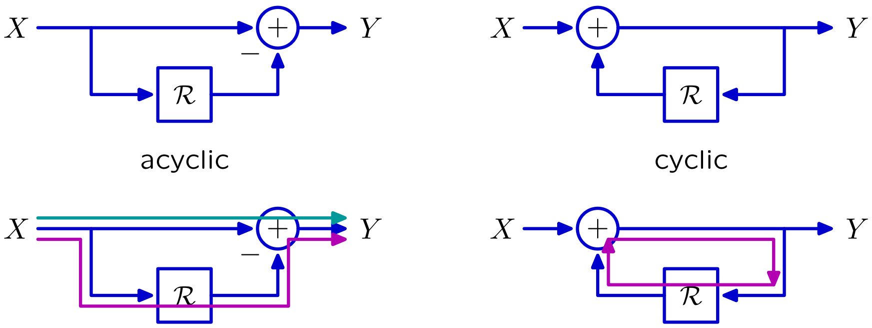
Feedforward and Feedback Systems

Feedforward systems that are constructed from adders, gains, and delays can be represented by polynomial operators of the form $\mathcal{F}(\mathcal{R})$.

Feedback systems are a bit more complicated because feedback systems contain cyclic signal-flow pathways.

Feedforward and Feedback Pathways

A **cyclic pathway** is one that closes a loop on itself.

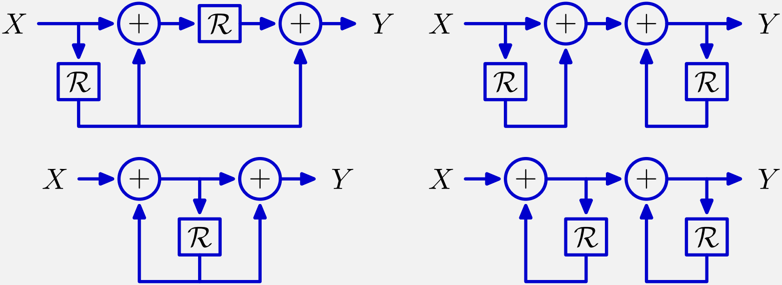


Feedforward systems contain no cyclic pathways. Their responses consist of a sum of components: each characterized by an aggregate gain and delay.

Feedback systems contain one or more cyclic pathways. Their responses can persist **long** after the input ends, as signals propagate through internal loops.

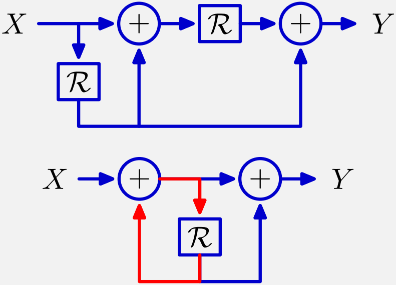
Check Yourself

How many of the following systems have cyclic signal paths?



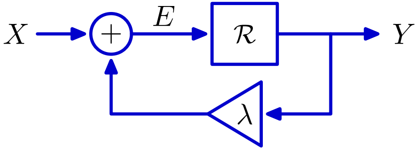
Check Yourself

How many of the following systems have cyclic signal paths? 3



System Functions for Feedback Systems

Determine Y when X is a geometric sequence: $X = z^n$.



$$E = X + \lambda Y$$

$$Y = \mathcal{R}E = \mathcal{R}X + \lambda \mathcal{R}Y$$

$$(1 - \lambda \mathcal{R})Y = \mathcal{R}X$$

Substitute z^n for X and z^{-1} for $\mathcal{R}$:

$$(1 - \lambda z^{-1})Y = z^{-1}z^n$$

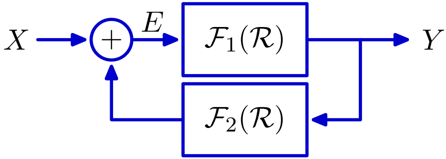
Assume that $1 - \lambda z^{-1}$ is a number that is not equal to zero.

Divide both sides by that number:

$$Y = \left(\frac{z^{-1}}{1 - \lambda z^{-1}}\right) z^n$$

System Functions for Feedback Systems

More generally, let $\mathcal{F}_1(\mathcal{R})$ represent the forward path and $\mathcal{F}_2(\mathcal{R})$ represent the feedback path.



$$Y = \mathcal{F}_1(\mathcal{R})E = \mathcal{F}_1(\mathcal{R})\Big(X + F_2(\mathcal{R})Y\Big) = \mathcal{F}_1(\mathcal{R})X + \mathcal{F}_1(\mathcal{R})F_2(\mathcal{R})Y$$

$$\Big(1 - \mathcal{F}_1(\mathcal{R})\mathcal{F}_2(\mathcal{R})\Big)Y = \mathcal{F}_1(\mathcal{R})X$$

Feedback introduces an operator expression on the left.

Substitute z^n for X and z^{-1} for $\mathcal{R}$:

$$\Big(1 - \mathcal{F}_1(z^{-1})\mathcal{F}_2(z^{-1})\Big)Y = \mathcal{F}_1(z^{-1})X$$

Assume that $1 - \mathcal{F}_1(z^{-1})\mathcal{F}_2(z^{-1})$ is a number that is not equal to zero.

Divide both sides by that number:

$$Y = \left(\frac{\mathcal{F}_1(z^{-1})}{1 - \mathcal{F}_1(z^{-1})\mathcal{F}_2(z^{-1})}\right)X = H(z)X \quad \text{where } H(z) = \frac{Y}{X}$$

Summary

Today we introduced **polynomial** (aka operator) representations of discrete time systems.

- The polynomial representation retains the structure of the underlying difference equations, and allows us to manipulate and simplify difference equations using **polynomial mathematics**.

- The polynomial representation facilitates the computation of responses to simple input signals such as **geometric signals:** z^{-1}.

- The polynomial representation provides a compact representation of a system in the form of a **system function** $H(z)$.

Next time: Using system functions to **analyze and design** control systems.