6.3100: Dynamic System Modeling and Control Design

DT System Functions and Poles

March 1, 2023

Last Time

We introduced an **operator** representation for discrete time systems.

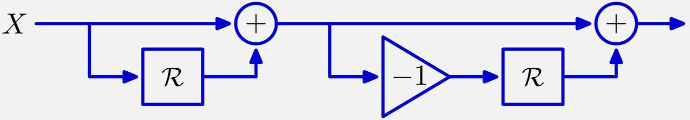
- The operator representation retains the structure of the underlying difference equations, and allows us to manipulate and simplify difference equations using **polynomial mathematics**.

- The polynomial representation facilitates the computation of responses to simple input signals such as **geometric signals:** z^{-1}.

- The polynomial representation provides a compact representation of a system in the form of a **system function** $H(z)$.

Check Yourself

Consider the following system.



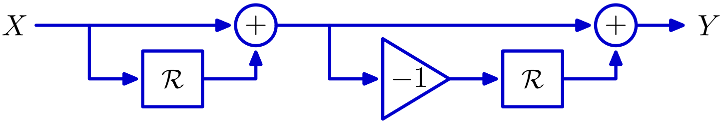
Find an operator $\mathcal{F}(\mathcal{R})$ so that $Y = \mathcal{F}(\mathcal{R})\{X\}$.

Use the operator $\mathcal{F}(\mathcal{R})$ to find the output Y_1 when X is $\left(\frac{1}{2}\right)^n$.

Use the operator $\mathcal{F}(\mathcal{R})$ to find the output Y_2 when X is z^n.

Check Yourself

Consider the following system.



Find an operator $\mathcal{F}(\mathcal{R})$ so that $Y = \mathcal{F}(\mathcal{R})\{X\}$.

$$Y = (1 - \mathcal{R})(1 + \mathcal{R})X = (1 - \mathcal{R}^2)X$$

Use the operator $\mathcal{F}(\mathcal{R})$ to find the output Y_1 when X is $\left(\frac{1}{2}\right)^n$.

$$Y = (1 - \mathcal{R}^2)\left\{\left(\tfrac{1}{2}\right)^n\right\} = \left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{2}\right)^{n-2} = \left(1 - \left(\tfrac{1}{2}\right)^{-2}\right)\left(\tfrac{1}{2}\right)^n = -3\left(\tfrac{1}{2}\right)^n$$

Use the operator $\mathcal{F}(\mathcal{R})$ to find the output Y_2 when X is z^n.

$$Y = (1 - \mathcal{R})(1 + \mathcal{R})X = (1 - \mathcal{R}^2)X$$

$$Y = (1 - \mathcal{R}^2)\{z^n\} = z^n - z^{n-2} = \left(1 - z^{-2}\right)z^n$$

Feedforward and Feedback Systems

Feedforward systems that are constructed from adders, gains, and delays can be represented by operator expressions of the form

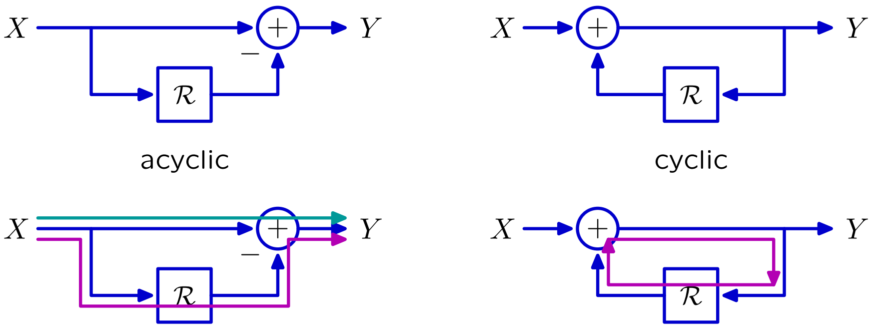
$$\mathcal{F}(\mathcal{R}) = a_0 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \cdots$$

where a_k are constants.

Feedback systems are a bit more complicated ...

Feedforward and Feedback Pathways

Feedforward and feedback systems differ in structure.

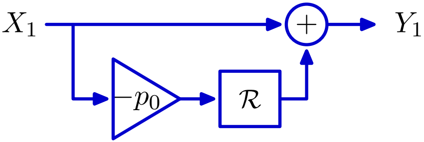


Feedforward systems contain no cyclic pathways. Their responses consist of a sum of components: each characterized by an aggregate gain and delay.

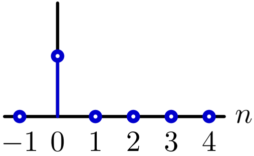
Feedback systems contain one or more cyclic pathways. Their responses can persist **long** after the input ends, as signals propagate through internal loops.

Transient and Persistent Responses

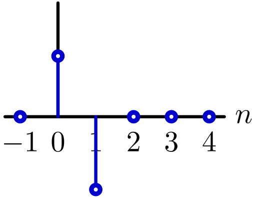
The following system is feedforward. It has no cyclic signal-flow pathways. Consider its response to a "unit-sample signal" $\delta[n]$.



$x_1[n] = \delta[n]$



$y_1[n] = x_1[n] - p_0\, x_1[n-1]$



The duration of its response to a unit-sample signal is limited by the highest order term in its operator representation:

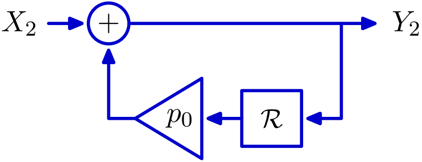
$$\mathcal{F}(\mathcal{R}) = 1 - p_0\mathcal{R}$$

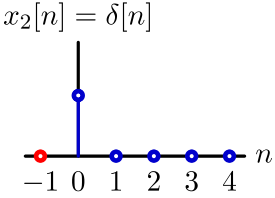
Transient and Persistent Responses

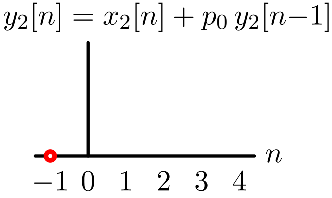
Systems with feedback can have **persistent** responses to **transient** inputs.

The following system has a cyclic signal-flow pathway.

Consider its response to a "unit-sample signal" $\delta[n]$.







Each cycle creates another sample in the output.

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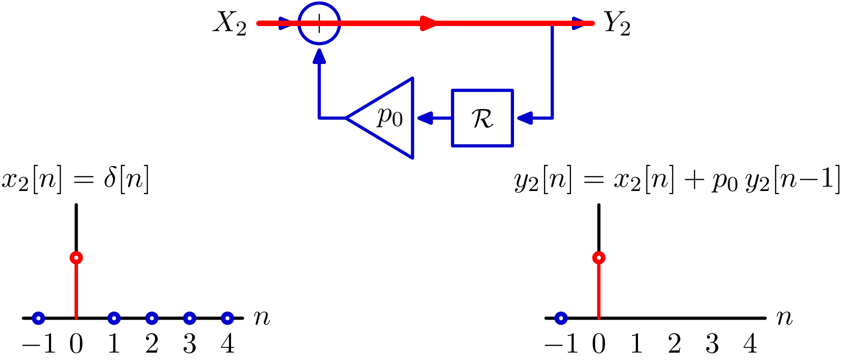

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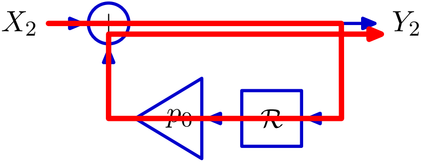

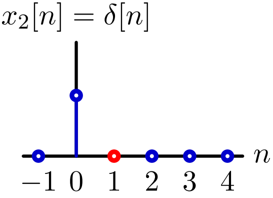

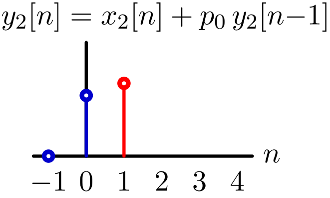

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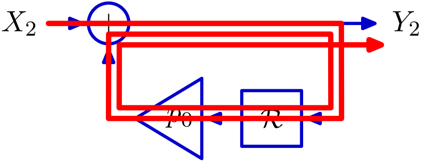
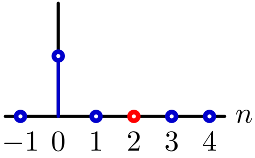
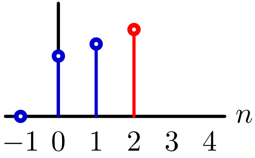


$x_2[n] = \delta[n]$



$y_2[n] = x_2[n] + p_0\, y_2[n-1]$



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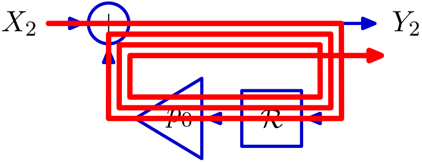

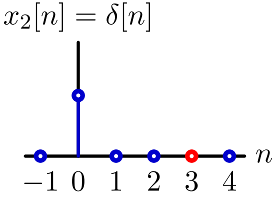

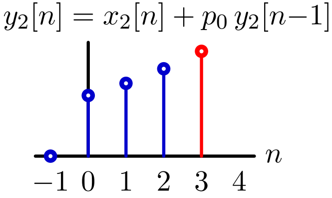

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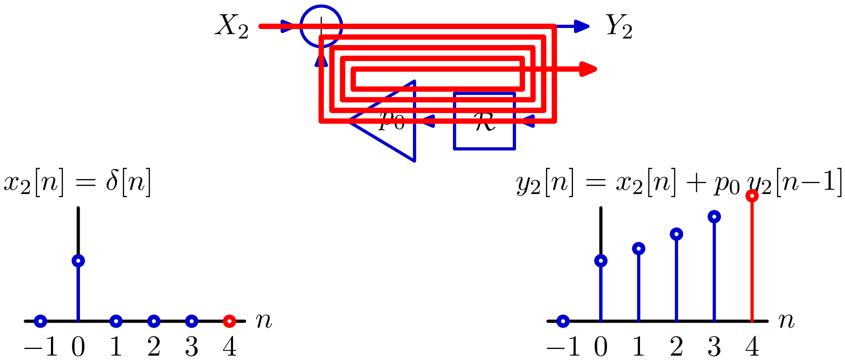

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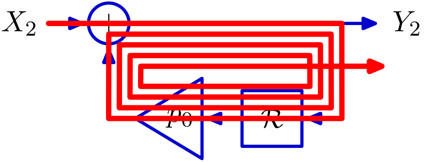
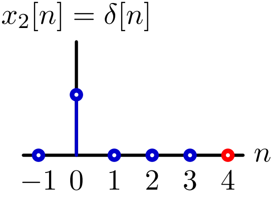
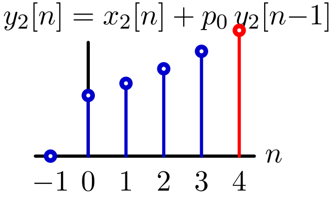


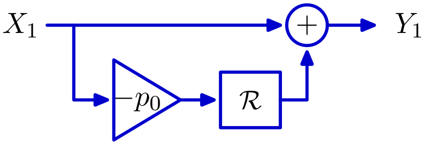


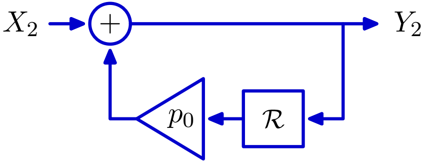


Each cycle creates another sample in the output.
The input X_2 is the shortest possible non-trivial signal ($\delta[n]$).
But the output Y_2 persists forever.
We say that this system has a **natural frequency** p_0.

Transient and Persistent Responses

Compare operator descriptions of these feedback and feedforward systems:





$$Y_1 = (1 - p_0\mathcal{R})X_1 \qquad\qquad (1 - p_0\mathcal{R})Y_2 = X_2$$

Assume that $X_1 = X_2 = z^n$.

Then we can find the outputs Y_1 and Y_2 by substituting z^{-1} for $\mathcal{R}$:

$$Y_1 = \left(1 - p_0 z^{-1}\right) z^n$$

$$(1 - p_0 z^{-1})Y_2 = z^n$$

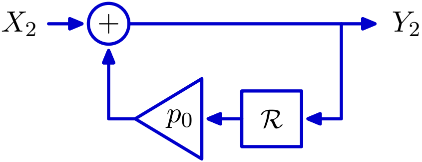
$$Y_2 = \left(\frac{1}{1 - p_0 z^{-1}}\right) z^n$$

$$H_1(z) = \frac{Y_1}{X_1} = 1 - p_0 z^{-1} \qquad\qquad H_2(z) = \frac{Y_2}{X_2} = \frac{1}{1 - p_0 z^{-1}}$$

These systems have **reciprocally related** system functions.

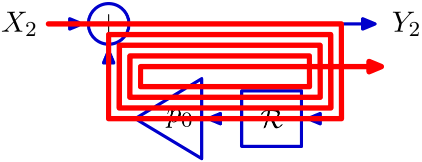
Transient and Persistent Responses

There are two equivalent operator representations for this feedback system. One follows directly from the block diagram:



$$(1 - p_0\mathcal{R})Y_2 = X_2$$

The other follows from tracing contributions by each pass through a cycle:



$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots)X_2$$

These expressions are equivalent in the sense that they both lead to the same output.

Transient and Persistent Responses

Compare the two operator representations for this feedback system.

Substitute the second:

$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots)X_2$$

into the first:

$$(1 - p_0\mathcal{R})Y_2 = X_2$$

to get the following:

$$(1 - p_0\mathcal{R})(1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots)X_2 = X_2$$

Multiplying:

$$\begin{aligned} &1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots \\ &\quad - p_0\mathcal{R} - p_0^2\mathcal{R}^2 - p_0^3\mathcal{R}^3 - p_0^4\mathcal{R}^4 - \cdots = 1 \end{aligned}$$

Therefore $(1-p_0\mathcal{R})$ and $(1+p_0\mathcal{R}+p_0^2\mathcal{R}^2+p_0^3\mathcal{R}^3+p_0^4\mathcal{R}^4+\cdots)$ are **reciprocals**.

We can think of the operator representation of this feedback system as

$$\mathcal{F}(\mathcal{R}) = \frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

Polynomial Interpretation of Reciprocals

The reciprocal relation between the two representations

$$\mathcal{F}(\mathcal{R}) = \frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

also follows from polynomial division.

$$\begin{array}{r|lllll}
 & 1 & +p_0\mathcal{R} & +p_0^2\mathcal{R}^2 & +p_0^3\mathcal{R}^3 & +\cdots \\ \hline
1 - p_0\mathcal{R} & 1 & & & & \\
 & 1 & -p_0\mathcal{R} & & & \\ \cline{2-3}
 & & p_0\mathcal{R} & & & \\
 & & p_0\mathcal{R} & -p_0^2\mathcal{R}^2 & & \\ \cline{3-4}
 & & & p_0^2\mathcal{R}^2 & & \\
 & & & p_0^2\mathcal{R}^2 & -p_0^3\mathcal{R}^3 & \\ \cline{4-5}
 & & & & p_0^3\mathcal{R}^3 & \\
 & & & & p_0^3\mathcal{R}^3 & -p_0^4\mathcal{R}^4 \\ \cline{5-6}
 & & & & & \cdots
\end{array}$$

This is another instance of how the normal rules of polynomial algebra apply to system operators.

System Functions

If the operator representation of a feedback system is

$$\mathcal{F}(\mathcal{R}) = \frac{1}{1 - p_0\mathcal{R}}$$

then the **system function** is

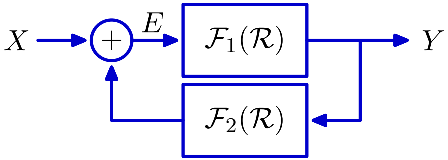
$$H(z) = \mathcal{F}(\mathcal{R})\Big|_{\mathcal{R}\to\frac{1}{z}} = \frac{1}{1 - p_0 z^{-1}} = \frac{z}{z - p_0}$$

The natural frequency p_0 is a root of the denominator of $H(z)$.

The roots of the denominator of a system function are called **poles**.

Black's Equation

More generally, let $\mathcal{F}_1(\mathcal{R})$ represent the forward path and $\mathcal{F}_2(\mathcal{R})$ represent the feedback path for a feedback system.



$$Y = \mathcal{F}_1(\mathcal{R})E = \mathcal{F}_1(\mathcal{R})\Big(X + F_2(\mathcal{R})Y\Big) = \mathcal{F}_1(\mathcal{R})X + \mathcal{F}_1(\mathcal{R})F_2(\mathcal{R})Y$$

$$\Big(1 - \mathcal{F}_1(\mathcal{R})\mathcal{F}_2(\mathcal{R})\Big)Y = \mathcal{F}_1(\mathcal{R})X$$

The transformation from X to Y is given by the operator expression

$$\mathcal{F}(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{F}_1(\mathcal{R})}{1 - \mathcal{F}_1(\mathcal{R})\mathcal{F}_2(\mathcal{R})}$$

and the system function is

$$H(z) = \frac{\mathcal{F}_1(z^{-1})}{1 - \mathcal{F}_1(z^{-1})\mathcal{F}_2(z^{-1})}$$

This equation is known as **Black's equation**.

Partial Fractions

The natural frequencys of a system can be identified by expanding the system functional $\mathcal{F}$ in partial fractions.

$$\mathcal{F} = \frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \cdots}$$

Factor denominator:

$$\mathcal{F} = \frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})(1 - p_2\mathcal{R})(1 - p_3\mathcal{R})\cdots}$$

Partial fractions:

$$\mathcal{F} = \frac{Y}{X} = \frac{C_0}{1 - p_0\mathcal{R}} + \frac{C_1}{1 - p_1\mathcal{R}} + \frac{C_2}{1 - p_2\mathcal{R}} + \cdots + D_0 + D_1\mathcal{R} + D_2\mathcal{R}^2 + \cdots$$

One natural frequency (p_i^n) arises from each factor of the denominator.

The polynomial terms (D_i) represent transient response components.

Poles

The form of each persistent mode is geometric, and the bases p_i of the geometrics are called the **poles** of the system.

$$\mathcal{F}(\mathcal{R}) = \frac{Y}{X} = \frac{C_0}{1-p_0\mathcal{R}} + \frac{C_1}{1-p_1\mathcal{R}} + \frac{C_2}{1-p_2\mathcal{R}} + \cdots + D_0 + D_1\mathcal{R} + D_2\mathcal{R}^2 + \cdots$$

Poles can be found by factoring the system functional $\mathcal{F}(\mathcal{R})$ as shown above. But an easier way to find the poles is to solve for the roots of the denominator of the system function $H(z)$:

$$H(z) = \mathcal{F}(\mathcal{R})\Big|_{\mathcal{R}\rightarrow\frac{1}{z}}$$

as shown below.

$$\begin{aligned} H(z) &= \frac{C_0}{1-p_0z^{-1}} + \frac{C_1}{1-p_1z^{-1}} + \frac{C_2}{1-p_2z^{-1}} + \cdots + D_0 + D_1z^{-1} + D_2z^{-2} + \cdots \\ &= \frac{C_0z}{z-p_0} + \frac{C_1z}{z-p_1} + \frac{C_2z}{z-p_2} + \cdots + D_0 + D_1z^{-1} + D_2z^{-2} + \cdots \end{aligned}$$

Check Yourself

Consider the system described by

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

How many of the following are true?

1. The unit sample response converges to zero.
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$.
3. There is a pole at $z = \frac{1}{2}$.
4. There are two poles.
5. None of the above

Check Yourself

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

$$\left(1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2\right)Y = \left(\mathcal{R} - \frac{1}{2}\mathcal{R}^2\right)X$$

$$H(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{R} - \frac{1}{2}\mathcal{R}^2}{1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2}$$

$$= \frac{\frac{1}{z} - \frac{1}{2}\frac{1}{z^2}}{1 + \frac{1}{4}\frac{1}{z} - \frac{1}{8}\frac{1}{z^2}} = \frac{z - \frac{1}{2}}{z^2 + \frac{1}{4}z - \frac{1}{8}} = \frac{z - \frac{1}{2}}{(z + \frac{1}{2})(z - \frac{1}{4})}$$

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$$= \frac{\frac{1}{z} - \frac{1}{2}\frac{1}{z^2}}{1 + \frac{1}{4}\frac{1}{z} - \frac{1}{8}\frac{1}{z^2}} = \frac{z - \frac{1}{2}}{z^2 + \frac{1}{4}z - \frac{1}{8}} = \frac{z - \frac{1}{2}}{(z + \frac{1}{2})(z - \frac{1}{4})}$$

1. The unit sample response converges to zero. ✓
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$. ✗
3. There is a pole at $z = \frac{1}{2}$. ✗
4. There are two poles. ✓
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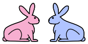
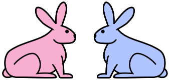
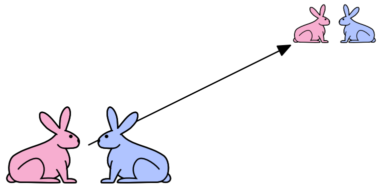
Population Growth



Population Growth



Population Growth



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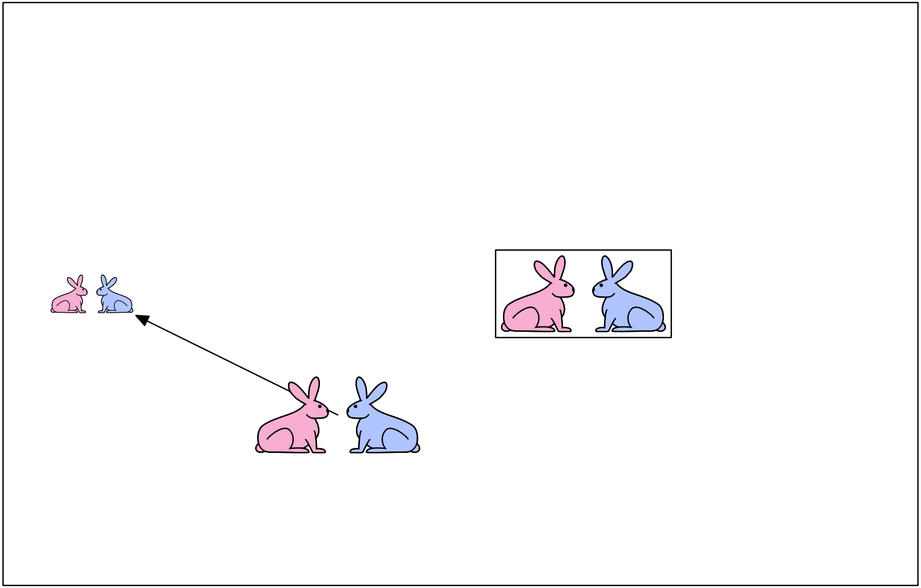

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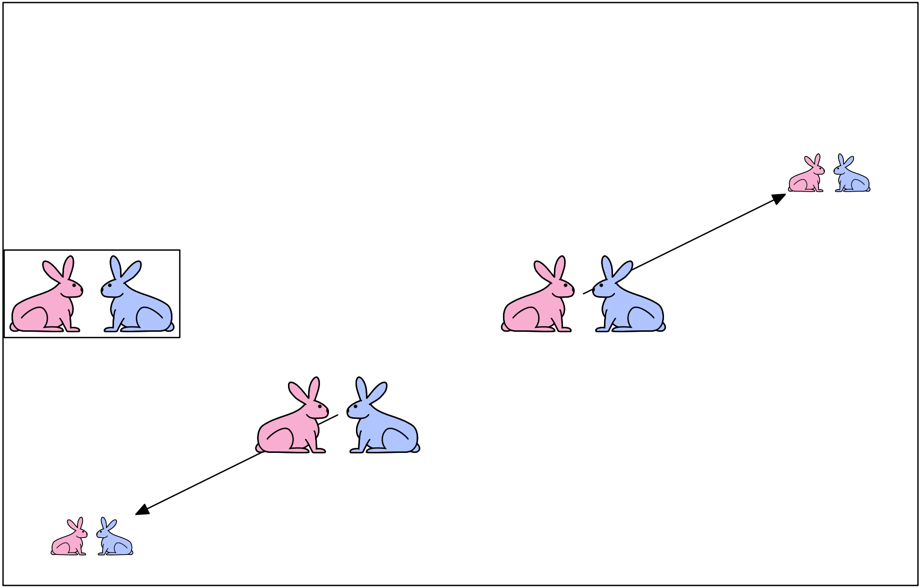

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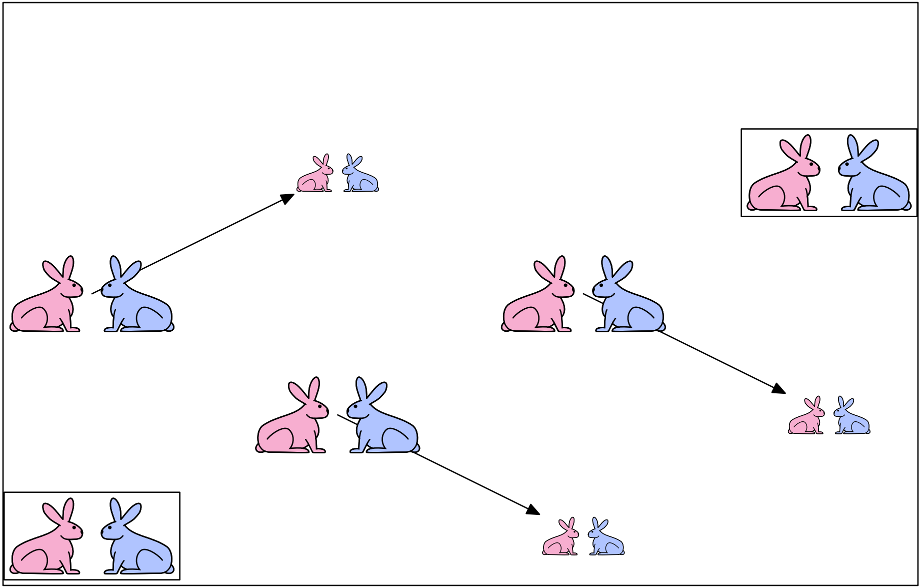

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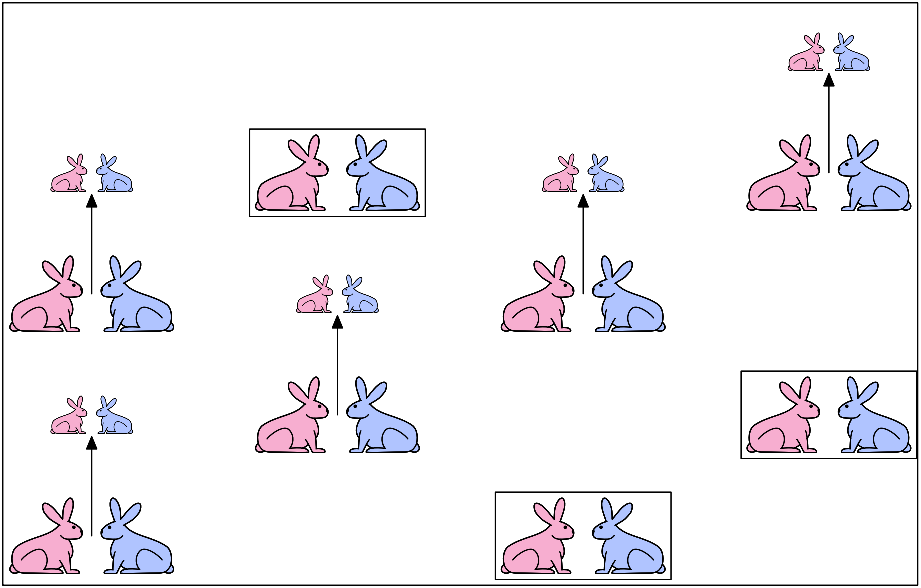
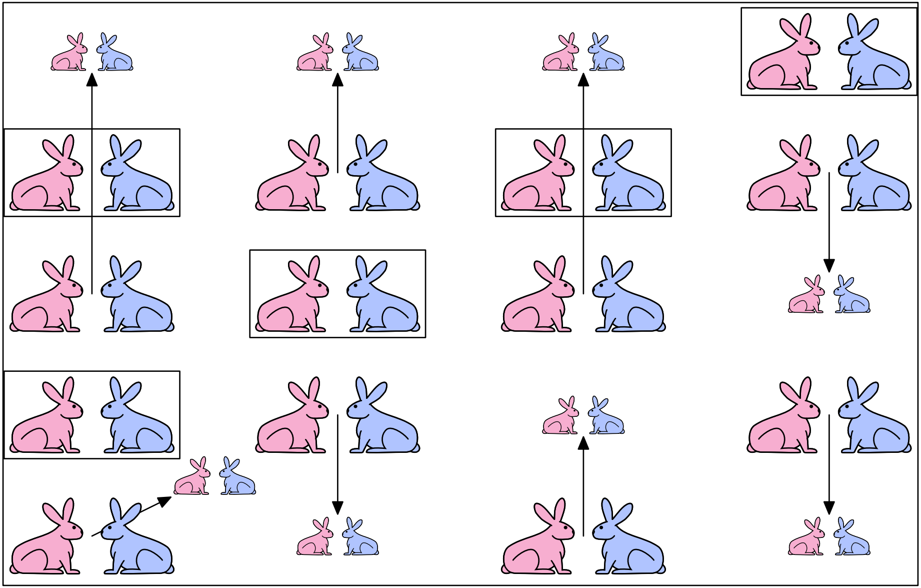
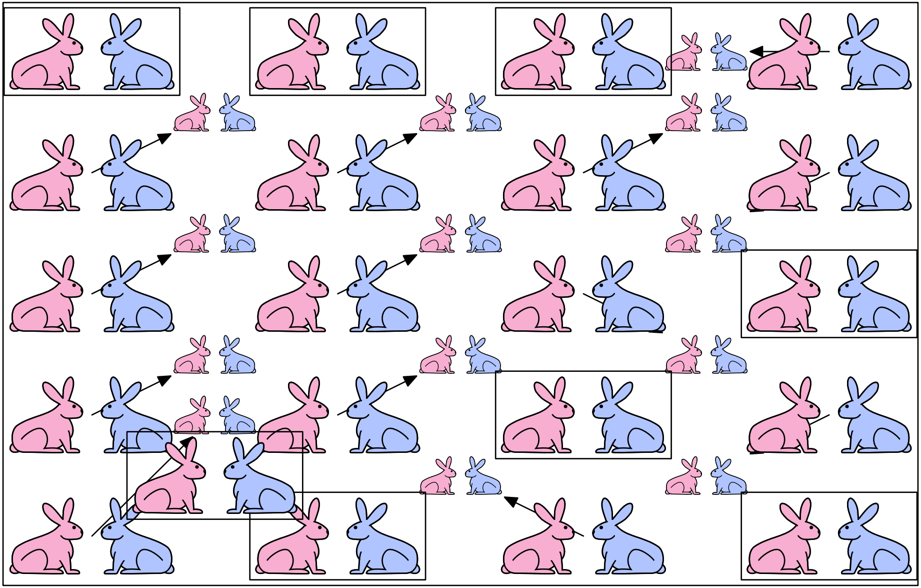


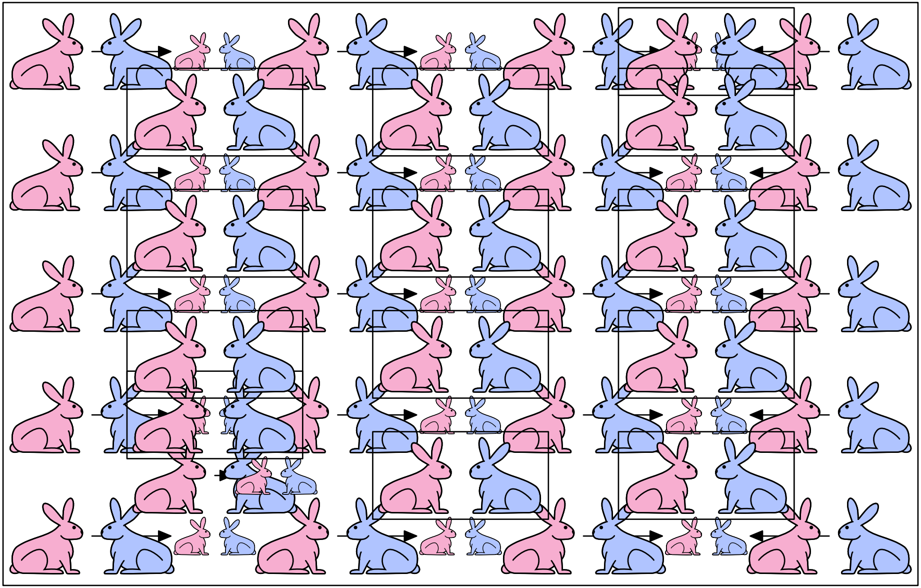
Population Growth



Population Growth



Population Growth



Fibonacci's Bunnies

Represent the number of pairs of bunnies in a population as a system:

where $x[n]$ represents the number of pairs of baby bunnies (children) added to the population in generation n and $a[n]$ represents the number of pairs of adults in generation n.

Let $c[n]$ represent the number of pairs of children in generation n. Assume that children become adults in one generation, so the total number of pairs of adults in generation n is the sum of the number of pairs of adults in generation $n-1$ plus the number of pairs of children in generation $n-1$.

$$a[n] = a[n-1] + c[n-1]$$

Each pair of adults produces a new pair of children in each generation, which adds to the number of pairs of children added externally ($x[n]$):

$$c[n] = x[n] + a[n-1]$$

Start the population by adding one pair of children at $n = 0$:

$$x[0] = 1$$

Check Yourself

What are the pole(s) of the bunnie system?

1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Check Yourself

What are the pole(s) of the bunnie system?

Difference equations for bunnie system:

$$a[n] = a[n-1] + c[n-1]$$

$$c[n] = x[n] + a[n-1]$$

$$a[n] = x[n-1] + a[n-1] + a[n-2]$$

System functional:

$$\mathcal{F}(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{R}}{1 - \mathcal{R} - \mathcal{R}^2}$$

System function:

$$H(z) = \frac{z}{z^2 - z - 1}$$

The denominator of the system function is second order $\rightarrow$ 2 poles.
The poles are at $z_1 = \frac{1+\sqrt{5}}{2} = 1.618$ and $z_2 = \frac{1-\sqrt{5}}{2} = -0.618$.

Check Yourself

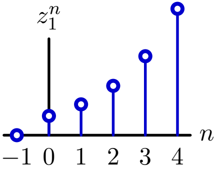
What are the pole(s) of the bunnie system? 4

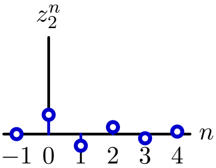
1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Example: Fibonacci's Bunnies

Each pole corresponds to a natural frequency.

$$z_1 \approx 1.618 \quad \text{and} \quad z_2 \approx -0.618$$





One mode diverges, one mode oscillates!

Summary

Today we characterized fundamental differences between feedforward and feedback systems.

- Feedforward systems can be characterized by a sum of components that are each characterized by an aggregate gain and delay.

- Feedback systems can be characterized by a ratio of polynomials in $\mathcal{R}$ or equivalently by a ratio of polynomials in z.

- The natural frequencies of a feedback system are given by its **poles**, which are the roots of the denominator of the system function.