6.3100: Dynamic System Modeling and Control Design

From Discrete-Time to Continuous-Time Systems

March 6, 2023

Overview

To date, we have focused on discrete-time systems.

- they are convenient and appropriate for microprocessor controllers
- DT systems are simpler to analyze than analogous CT systems

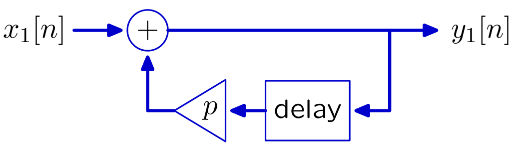
But some issues (such as frequency responses) are easier to analyze using continuous-time methods.

Fortunately, methods and insights from discrete-time design and analysis are (for the most part) easily translated to continuous time.

Comparison of Discrete-Time and Continuous-Time Systems

discrete-time

block diagrams with delays

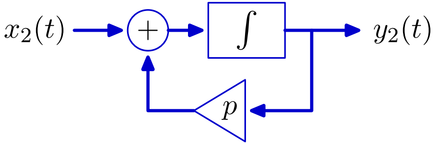


difference equation

$$y_1[n] = x_1[n] + py_1[n-1]$$

continuous-time

block diagrams with integrators



differential equation

$$\frac{dy_2(t)}{dt} = x_2(t) + py_2(t)$$

Determine the **homogeneous solutions** when $y_1[0] = 1$ and $y_2(0) = 1$.

For what value(s) of p do the homogeneous solutions **converge/diverge**?

Discrete-Time Homogeneous Solution

Determine the response $y_1[n]$ in the following difference equation

$$y_1[n] = x_1[n] + py_1[n-1]$$

when $x_1[n] = 0$ for all n and $y_1[0] = 1$.

Since the input is zero, we expect that the output will be a natural frequency with the following geometric form:

$$y_1[n] = Cz^n$$

where C and z are constants. Substituting into the difference equation

$$Cz^n = 0 + pCz^{-1}z^n$$

so $z = p$. We can find C from the initial condition

$$y_1[0] = 1 = Cz^0 = C$$

and the final answer is

$$y_1[n] = p^n \text{ for } n \geq 0$$

Continuous-Time Homogeneous Solution

Determine the response $y_2(t)$ in the following difference equation

$$\frac{dy_2(t)}{dt} = x(t) + py(t)$$

when $x_2(t) = 0$ for all t and $y_2(0) = 1$.

Since the input is zero, we expect that the output will be a natural frequency with the following exponential form:

$$y_2(t) = Ce^{st}$$

where C and s are constants. Substituting into the difference equation

$$Cse^{st} = 0 + pCe^{st}$$

so $s = p$. We can find C from the initial condition

$$y_2(0) = 1 = Ce^{st} = C$$

and the final answer is

$$y_2(t) = e^{pt} \text{ for } t \geq 0$$

Discrete-Time Convergence

For what values of p does the sequence p^n converge as $n \to \infty$?

$$\lim_{n\to\infty} p^n = ?$$

Express p in polar coordinates:

$$p = Me^{j\phi}$$

so that

$$\lim_{n\to\infty} p^n = \lim_{n\to\infty} \left(Me^{j\phi}\right)^n = \lim_{n\to\infty} M^n e^{j\phi n}$$

The CT homogeneous solution converges to 0 if $M = |p| < 1$.

Continuous-Time Convergence

For what values of p does the function e^{pt} converge as $t \to \infty$?

$$\lim_{t\to\infty} e^{pt} = ?$$

Express p in rectangular coordinates:

$$p = a + jb$$

so that

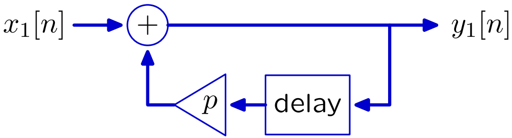
$$\lim_{t\to\infty} e^{pt} = \lim_{t\to\infty} e^{(a+jb)t} = \lim_{t\to\infty} e^{at} e^{jbt}$$

The DT homogeneous solution converges to 0 if $a = \mathrm{Re}\,(p) < 0$.

Comparison of Discrete-Time and Continuous-Time Systems

discrete-time

block diagrams with delays



difference equation

$$y_1[n] = x_1[n] + py_1[n-1]$$

homogeneous solution

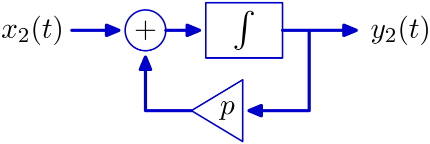
$$y_1[n] = p^n; \quad n \geq 0$$

region of convergence

$$|p| < 1$$

continuous-time

block diagrams with integrators



differential equation

$$\frac{dy_2(t)}{dt} = x_2(t) + py_2(t)$$

homogeneous solution

$$y_2(t) = e^{pt} \quad t \geq 0$$

region of convergence

$$\mathrm{Re}(p) < 0$$

Many similarities. Some differences.

Comparison of Higher-Order Systems

difference equation

$$y_1[n] = x_1[n] + y_1[n-1] + y_1[n-2]$$

differential equation

$$\frac{d^2y_2(t)}{dt^2} = x_2(t) + \frac{dy_2(t)}{dt} + y_2(t)$$

system function

$$H_1(z) = \frac{Y_1}{X_1}$$

system function

$$H_2(s) = \frac{Y_2}{X_2}$$

Determine system functions $H_1(z)$ and $H_2(s)$ for these higher-order systems.

System Functions

Start with the discrete-time system function.

If $y_1[n]$ is geometric with the form $y_1[n] = z^n$, then right-shifting n by 1 multiplies the signal by z^{-1} and right-shifting n by 2 multiplies the signal by z^{-2}: Substitute these relations into the difference equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

to get

$$Y = X + z^{-1}Y + z^{-2}Y$$

Solve for the ratio of Y to X:

$$H_1(z) = \frac{Y_1}{X_1} = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$$

System Functions

We can do the same sort of analysis for the continuous-time system.

If $y_2(t)$ is exponential with the form $y_2(t) = e^{st}$, then differentiating multiplies by s. Substitute this relation into the differential equation

$$\frac{d^2y_2(t)}{dt^2} = x_2(t) + \frac{dy_2(t)}{dt} + y_2(t)$$

to get

$$s^2Y_2 = X_2 + sY_2 + Y_2$$

Solve for the ratio of Y_2 to X_2:

$$H_2(s) = \frac{Y_2}{X_2} = \frac{1}{s^2 - s - 1}$$

System Functions

difference equation

$$y_1[n] = x_1[n] + y_1[n-1] + y_1[n-2]$$

system function

$$H_1(z) = \frac{z^2}{z^2 - z - 1}$$

differential equation

$$\frac{d^2 y_2(t)}{dt^2} = x_2(t) + \frac{dy_2(t)}{dt} + y_2(t)$$

system function

$$H_2(s) = \frac{1}{s^2 - s - 1}$$

Both DT and CT system functions are **ratios of polynomials**, which can be found from their responses to their respective **eigenfunctions**:

- z^n for DT systems
- e^{st} for CT systems

Comparison of Higher-Order Systems

difference equation

$$y_1[n] = x_1[n] + y_1[n-1] + y_1[n-2]$$

system function

$$H_1(z) = \frac{z^2}{z^2 - z - 1}$$

differential equation

$$\frac{d^2y_2(t)}{dt^2} = x_2(t) + \frac{dy_2(t)}{dt} + y_2(t)$$

system function

$$H_2(s) = \frac{1}{s^2 - s - 1}$$

Determine the **natural frequencies** of these systems and their corresponding time functions (aka **fundamental modes**).

System Functions

Start with the discrete-time system function.

$$H_1(z) = \frac{Y_1}{X_1} = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$$

The denominator of the system function is second order and has two roots

$$z_1, z_2 = \tfrac{1}{2} \pm \sqrt{\left(\tfrac{1}{2}\right)^2 + 1} = \tfrac{1}{2}(1 \pm \sqrt{5})$$

which are the two natural frequencies.

The corresponding time waveforms (fundamental modes) are

$$C_1 z_1^n$$

and

$$C_2 z_2^n$$

where C_1 and C_2 are constants that are determined by initial conditions.

System Functions

We can do the same sort of analysis for the continuous-time system.

$$H_2(s) = \frac{Y_2}{X_2} = \frac{1}{s^2 - s - 1}$$

The denominator of the system function is second order and has two roots

$$s_1, s_2 = \tfrac{1}{2} \pm \sqrt{\left(\tfrac{1}{2}\right)^2 + 1} = \tfrac{1}{2}(1 \pm \sqrt{5})$$

which are the two natural frequencies.

The corresponding time waveforms (fundamental modes) are

$$C_1 e^{s_1 t}$$

and

$$C_2 e^{s_2 t}$$

where C_1 and C_2 are constants that are determined by initial conditions.

System Functions

difference equation

$$y_1[n] = x_1[n] + y_1[n-1] + y_1[n-2]$$

system function

$$H_1(z) = \frac{z^2}{z^2 - z - 1}$$

poles

$$z_1, z_2 = \tfrac{1}{2}(1 \pm \sqrt{5})$$

homogeneous solutions

$$C_1 z_1^n + C_2 z_2^n$$

differential equation

$$\frac{d^2 y_2(t)}{dt^2} = x_2(t) + \frac{dy_2(t)}{dt} + y_2(t)$$

system function

$$H_2(s) = \frac{1}{s^2 - s - 1}$$

poles

$$s_1, s_2 = \tfrac{1}{2}(1 \pm \sqrt{5})$$

homogeneous solutions

$$C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

- **poles** are the roots of the denominator of the system function
- each pole corresponds to a **natural frequency**
- homogeneous solution is a sum of contributions from each pole

Complex Poles

Oscillatory responses of both discrete-time and continuous-time systems result when a pole has a non-zero imaginary part.

Fundamental modes for discrete-time systems have the form

$$p^n = \left(Me^{j\phi}\right)^n$$

and responses will be monotonic if $\phi = 0$ or alternating if $\phi = \pi$.

Fundamental modes for continuous-time systems have the form

$$e^{pt} = e^{(a+jb)t} = e^{at}e^{-jbt}$$

and oscillations will occur if $b \neq 0$.

Check Yourself

Match the system functions on the left with the poles on the right.

A: $\frac{1}{1-\mathcal{R}-\mathcal{R}^2}$

B: $\frac{1}{1+\mathcal{R}-\mathcal{R}^2}$

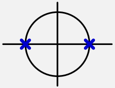
C: $\frac{1}{1-\mathcal{R}+\mathcal{R}^2}$

D: $\frac{1}{1+\mathcal{R}+\mathcal{R}^2}$

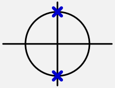
E: $\frac{1}{1+\mathcal{R}^2}$

F: $\frac{1}{1-\mathcal{R}^2}$

a



b



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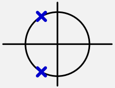

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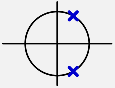

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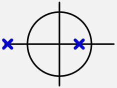

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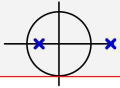


Check Yourself

Match the system functions on the left with the poles on the right.

A: $\frac{1}{1-\mathcal{R}-\mathcal{R}^2}$

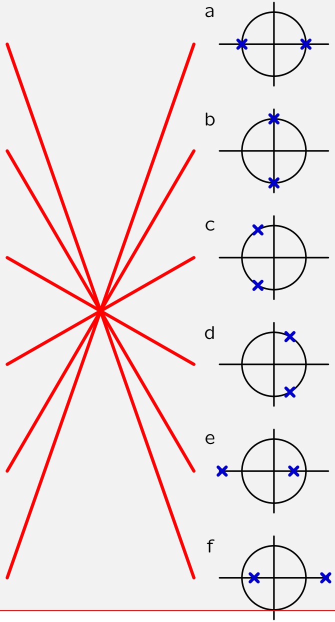
B: $\frac{1}{1+\mathcal{R}-\mathcal{R}^2}$

C: $\frac{1}{1-\mathcal{R}+\mathcal{R}^2}$

D: $\frac{1}{1+\mathcal{R}+\mathcal{R}^2}$

E: $\frac{1}{1+\mathcal{R}^2}$

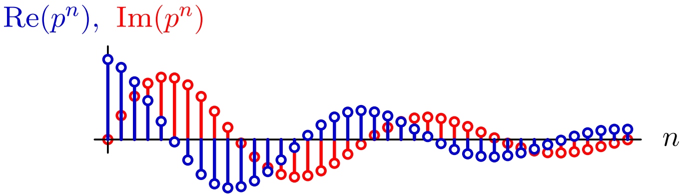
F: $\frac{1}{1-\mathcal{R}^2}$



Complex Poles in DT systems

The fundamental mode associated with a complex pole has both real and imaginary parts.

Example: A pole at $p = 0.9 + j0.3$ generates a geometric sequence p^n:



But the responses of physical systems are real!

The pole in this example comes from the following system:

$$y[n] = x[n] + 1.8y[n-1] - 0.9y[n-2]$$

If $x[n]$ is real for all n, and if the initial conditions are also real, then $y[n]$ should be real.

What's going on?

Complex Poles

If $x[n]$ and $y[n]$ represent real-world signals, then the coefficients of the numerator and denominator of the system function are real.

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \cdots}$$

Factor theorem: a polynomial can be expressed as a product of factors, with one factor associated with each root of the polynomial.

Fundamental theorem of algebra: a polynomial of order n has n roots. The roots can have imaginary parts.

Factor denominator:

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})(1 - p_2\mathcal{R})(1 - p_3\mathcal{R})\cdots}$$

Partial fractions:

$$\frac{Y}{X} = \frac{C_0}{1 - p_0\mathcal{R}} + \frac{C_1}{1 - p_1\mathcal{R}} + \frac{C_2}{1 - p_2\mathcal{R}} + \cdots + D_0 + D_1\mathcal{R} + D_2\mathcal{R}^2 + \cdots$$

Complex Roots

If p is a root of a polynomial with constant real-valued coefficients, then its complex-conjugate p^* is also a root.

Proof. Let $D(z)$ represent a polynomial in z with constant real-valued coefficients.

If p is a root of $D(z)$ then $D(p) = 0$.

Since all of the coefficients are real-valued,

$$D(p^*) = (D(p))^* = 0^* = 0.$$

Thus p^* is also a root.

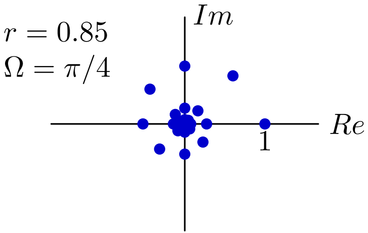
If we **pair** the factor associated with $p{=}a{+}jb$ with the factor associated with $p^*{=}a{-}jb$, we get a second-order polynomial with **real** coefficients:

$$(z{-}p)(z{-}p^*) = (z{-}a{-}jb)(z{-}a{+}jb) = z^2{-}2az{+}a^2{+}b^2$$

Complex Poles

A complex-valued pole produces a complex-valued fundamental mode.

$$p^n = \left(Me^{j\phi}\right)^n$$



The magnitude M^n grows geometrically with n, and the angle ϕn grows linearly with n.

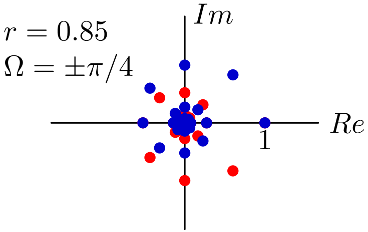
Complex Poles

The fundamental mode associated with the complex conjugate of p

$$(p^*)^n = \left(Me^{-j\phi}\right)^n$$

has the same magnitude as that for p and opposite angle.

$$p^n = \left(Me^{j\phi}\right)^n$$



The sum of the fundamental modes associated with p and p^* is real-valued.

Complex Roots

An isolated complex root can only result if the difference equation has one or more complex-valued coefficients.

Example:

$$\frac{Y}{X} = \frac{1}{1 - (a+jb)\mathcal{R}}$$

Corresponding difference equation:

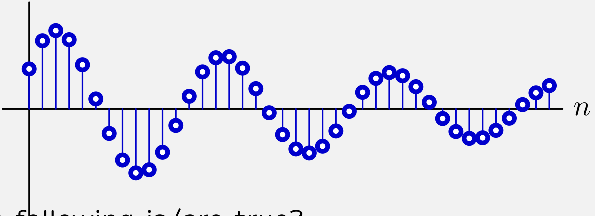
$$y[n] - (a+jb)\, y[n-1] = x[n]$$

A first-order system that represents a physical system can only have real-valued poles

A second-order or higher-order system that represents a physical system can have poles with imaginary parts, but such poles occur in **complex conjugate pairs**.

Check Yourself

Unit-sample response of a system with poles at $z = re^{\pm j\Omega}$.

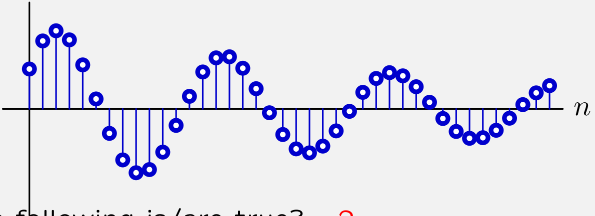


Which of the following is/are true?

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Check Yourself

Unit-sample response of a system with poles at $z = re^{\pm j\Omega}$.



Which of the following is/are true? 2

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Summary

Methods and insights from discrete-time design and analysis are (for the most part) easily translated to continuous time.

Fundamental modes for DT have the form z^n.
Fundamental modes for CT have the form e^{st}.

DT system functions are rational polynomials in z.
CT system functions are rational polynomials in s.

The poles of both DT and CT systems are the roots of the denominator of the system function.

DT systems are stable if all of the poles are inside the unit circle.
CT systems are stable if all of the poles are in the left half-plane.