6.3100: Dynamic System Modeling and Control Design

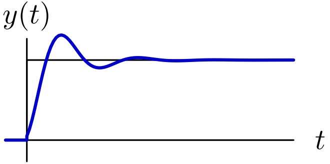
Frequency Response

March 8, 2023

From Transients to Frequency Responses

To date, we have described systems by their responses to sudden changes in their input.

Example: step response

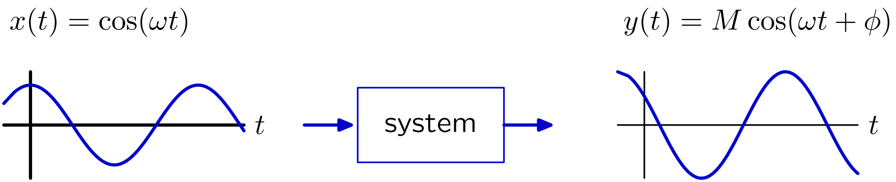


Today we will look at a different (but mathematically equivalent) characterization based on sinusoids – the **frequency response**.

Frequency Response Preview

If the input/output relation of a system can be described by a linear differential equation with constant coefficients, then its response to a sinusoid will be a sinusoid with

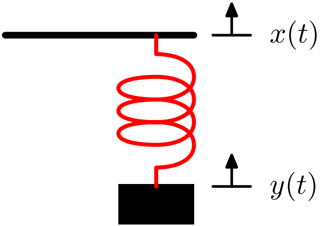
- the same frequency,
- possibly different amplitude, and
- possibly different phase angle.

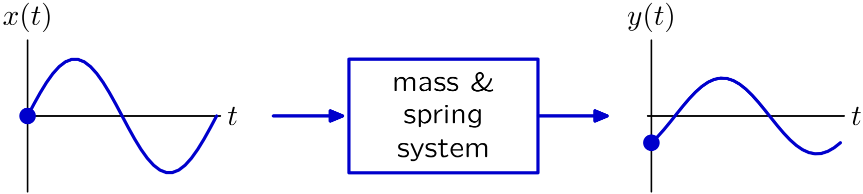


The **frequency response** is a plot of the magnitude M and angle ϕ as a function of $\omega = 2\pi f$ where f is the frequency in Hertz (cycles/second).

- natural way to describe many systems and disturbances
- new way to think about the design of control systems (next week)

Example: Mass and Spring





At low frequencies, the output is approximately equal to the input.

At middle frequencies, the output can get very large. There is a **resonance**.

At high frequencies, the output is small.

Frequency Response Calculation

A straightforward way to compute a frequency response is to substitute

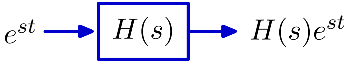
$$x(t) = \cos(\omega t)$$

into the system's differential equation and solve for the response $y(t)$.

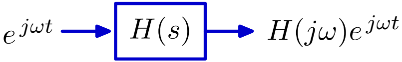
But there are a number of much easier methods based on our work with eigenfunctions and system (transfer) functions.

System Function Approach

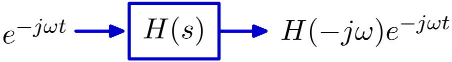
Start with the definition of the system function as the eigenvalue associated with the eigenfunction e^{st}.



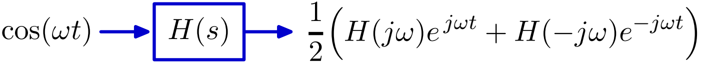
Since s represents an arbitrary complex number, we can subsitute $j\omega$ for s:



We can similarly substitute $-j\omega$ for s:



and then use Euler's formula to determine the response to a cosine:



This expression can be simplified when $H(s)$ is the ratio of polynomials with real-valued coefficients.

Real-Valued System Functions

If a system can be represented by a linear differential equation with constant, **real-valued** coefficents:

$$\sum_k a_k \frac{d^k y(t)}{dt^k} = \sum_k b_k \frac{d^k x(t)}{dt^k}$$

then the system function can be represented as the ratio polynomials in s whose coefficients are real-valued.

$$H(s) = \frac{\sum_k a_k s^k}{\sum_k b_k s^k}$$

System Function Approach

Simplifying the expression for the response to a cosine input.

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow \frac{1}{2}\Big(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\Big)$$

If $x(t) = \cos(\omega t)$ then

$$\begin{aligned} y(t) &= \frac{1}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right) \\ &= \mathrm{Re}\left\{H(j\omega)e^{j\omega t}\right\} \\ &= \mathrm{Re}\left\{|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}\right\} \\ &= |H(j\omega)|\mathrm{Re}\left\{e^{j\omega t + j\angle H(j\omega)}\right\} \end{aligned}$$

$$y(t) = |H(j\omega)|\cos\left(\omega t + \angle H(j\omega)\right).$$

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)|\cos\left(\omega t + \angle H(j\omega)\right)$$

The frequency response is equal to the **magnitude and angle** of the system function $H(s)$ evaluated at $s = j\omega$: $H(s)\Big|_{s=j\omega}$

Check Yourself

Compare two methods for determinining the magnitude and angle of the frequency response of the system described by the following differential equation:

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} - y(t)$$

Method 1: solve the differential equation

Method 2: find the magnitude and angle of $H(j\omega)$

Check Yourself

Find the magnitude and angle of the frequency response of the system described by the following differential equation:

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} - y(t)$$

by solving the differential equation.

$$x(t) = \cos(\omega t)$$

$$\frac{dx(t)}{dt} = -\omega \sin(\omega t)$$

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\frac{dy(t)}{dt} = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$$

$$-A\omega \sin(\omega t) + B\omega \cos(\omega t) = -\omega \sin(\omega t) - A\cos(\omega t) - B\sin(\omega t)$$

$$-A\omega = -\omega - B \quad \text{and} \quad B\omega = -A$$

$$y(t) = \frac{\omega^2}{1+\omega^2}\cos(\omega t) - \frac{\omega}{1+\omega^2}\sin(\omega t)$$

Now convert to magnitude and angle ... too complicated!

Check Yourself

Use Euler's formula!

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} - y(t)$$

$$x(t) = \text{Re}\left(e^{j\omega t}\right)$$

$$\frac{dx(t)}{dt} = \text{Re}\left(j\omega e^{j\omega t}\right)$$

$$y(t) = \text{Re}\left(Ce^{j\omega t}\right)$$

$$\frac{dy(t)}{dt} = \text{Re}\left(j\omega Ce^{j\omega t}\right)$$

$$j\omega Ce^{j\omega t} = j\omega e^{j\omega t} - Ce^{j\omega t}$$

$$j\omega C = j\omega - C$$

$$C = \frac{j\omega}{1 + j\omega}$$

$$|C|^2 = \frac{\omega^2}{1 + \omega^2}$$

$$\angle(C) = \frac{\pi}{2} - \tan^{-1}(\omega)$$

Check Yourself

Evaluate the system function $H(s)$ at $x = j\omega$.

$$\frac{dy(t)}{dt} = \frac{dx(t)}{dt} - y(t)$$

$$sY = sX - Y$$

$$H(s) = \frac{Y}{X} = \frac{s}{1+s}$$

$$H(j\omega) = \frac{j\omega}{1+j\omega}$$

$$|H(j\omega)|^2 = \frac{\omega^2}{1+\omega^2}$$

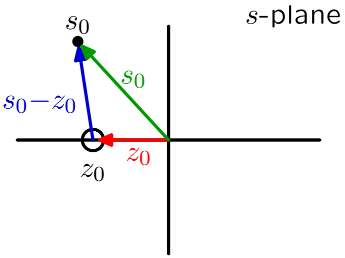
$$\angle(H(j\omega)) = \frac{\pi}{2} - \tan^{-1}(\omega)$$

Vector Diagrams

The value of $H(s)$ at a point $s{=}s_0$ can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K\frac{(s_0-z_0)(s_0-z_1)(s_0-z_2)\cdots}{(s_0-p_0)(s_0-p_1)(s_0-p_2)\cdots}$$



Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here z_0) to s_0, the point of interest in the s-plane.

Vector Diagrams

The value of $H(s)$ at a point $s{=}s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K\frac{(s_0{-}z_0)(s_0{-}z_1)(s_0{-}z_2)\cdots}{(s_0{-}p_0)(s_0{-}p_1)(s_0{-}p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(s_0)| = |K|\frac{|(s_0{-}z_0)||(s_0{-}z_1)||(s_0{-}z_2)|\cdots}{|(s_0{-}p_0)||(s_0{-}p_1)||(s_0{-}p_2)|\cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle(s_0{-}z_0) + \angle(s_0{-}z_1) + \cdots - \angle(s_0{-}p_0) - \angle(s_0{-}p_1) - \cdots$$

Vector Diagrams

The frequency response is equal to $H(s)$ at $s=j\omega$.

The value of $H(s)$ at a point $s=j\omega$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(j\omega) = K\frac{(j\omega-z_0)(j\omega-z_1)(j\omega-z_2)\cdots}{(j\omega-p_0)(j\omega-p_1)(j\omega-p_2)\cdots}$$

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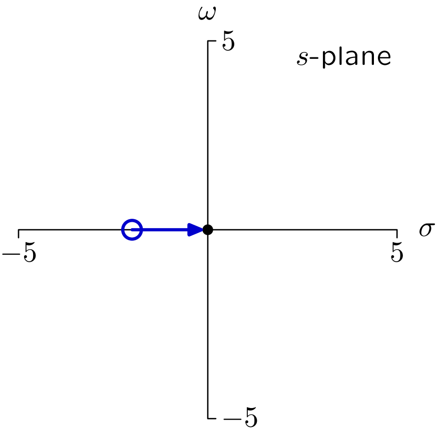
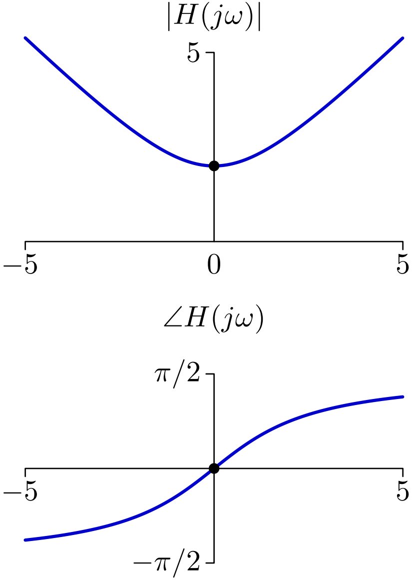
Vector Diagrams

$$H(s) = s - z_1$$





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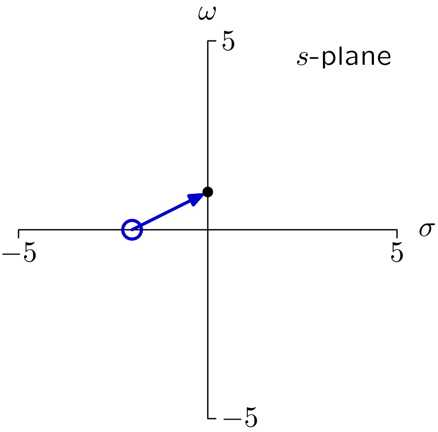

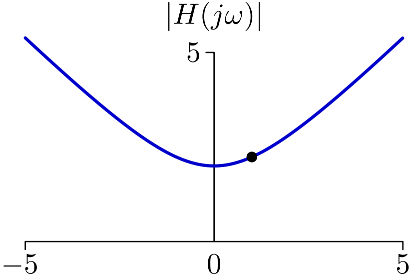

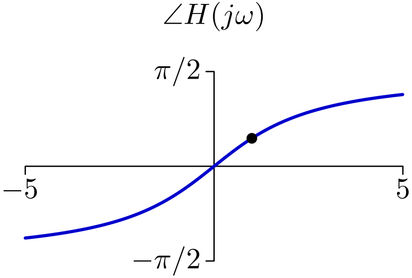

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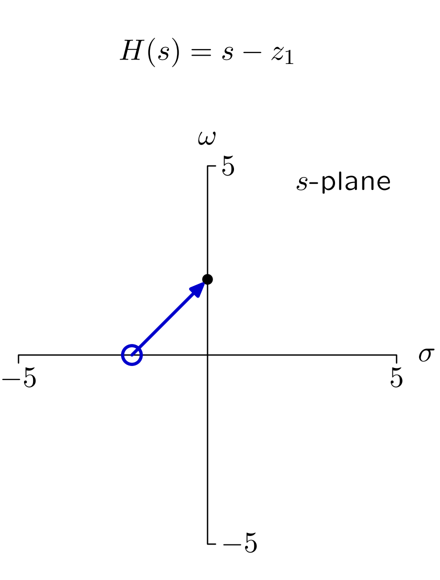

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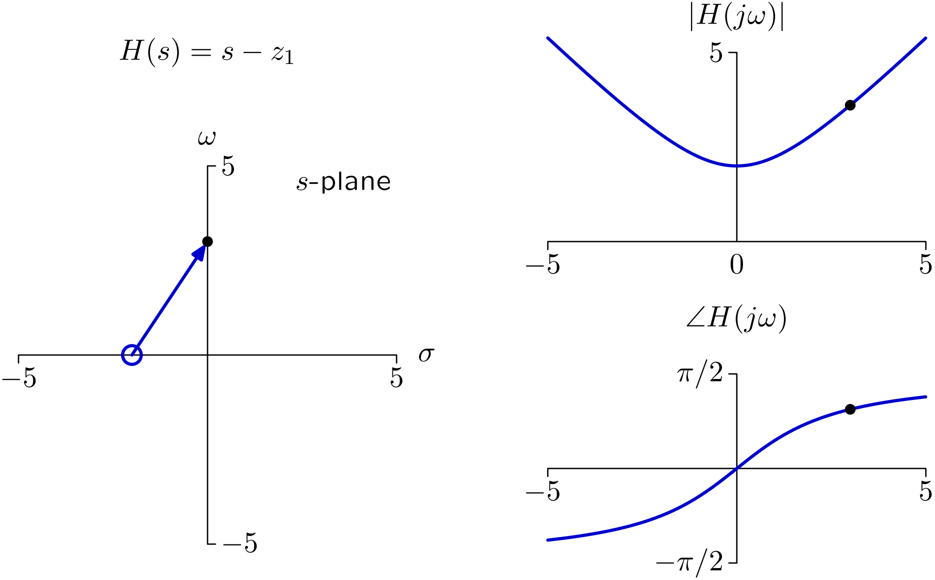

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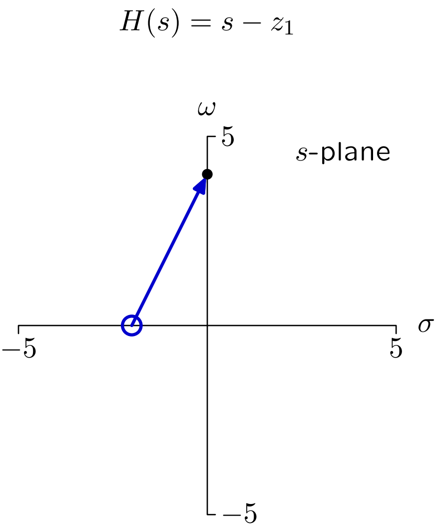

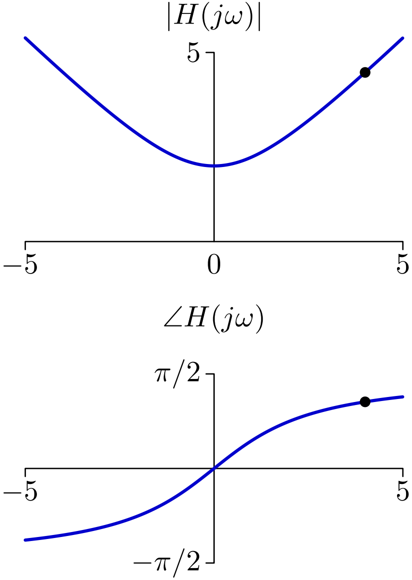

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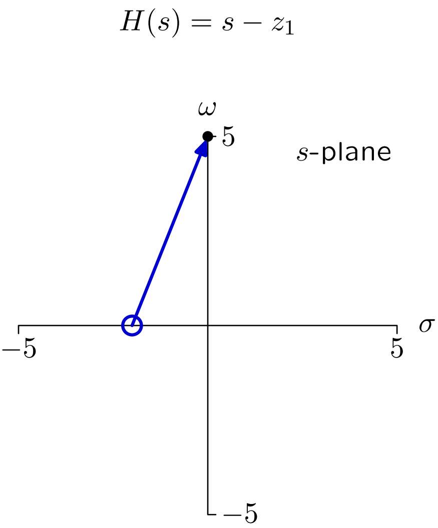

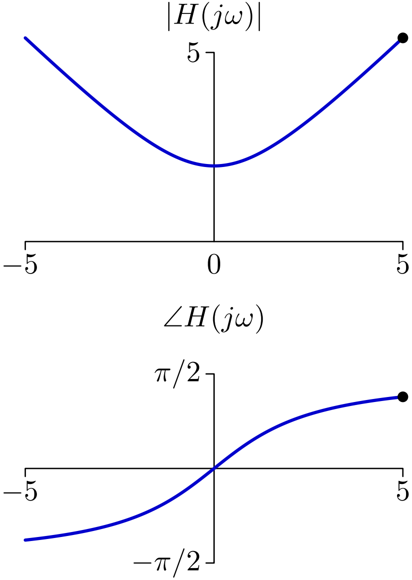

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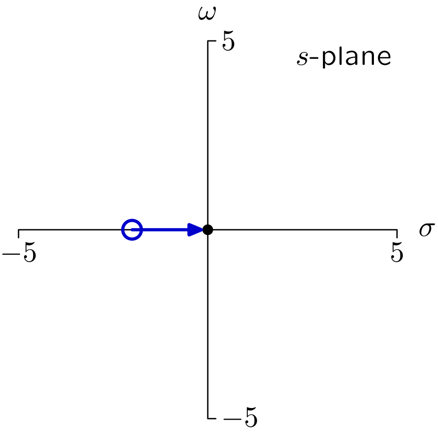

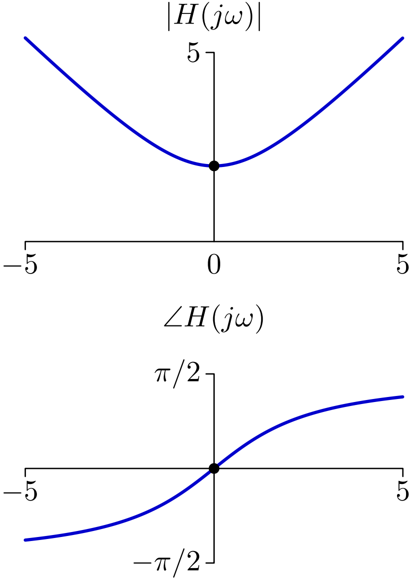

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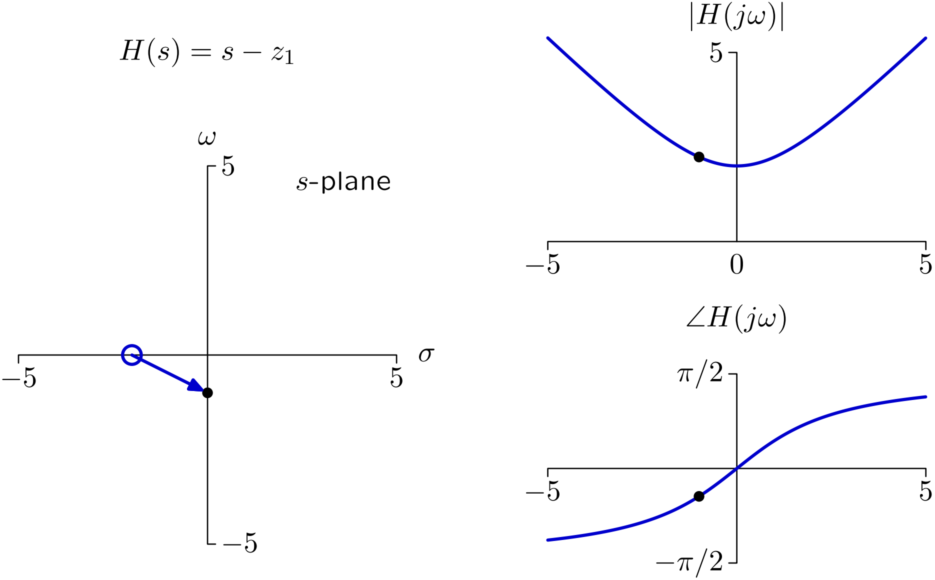

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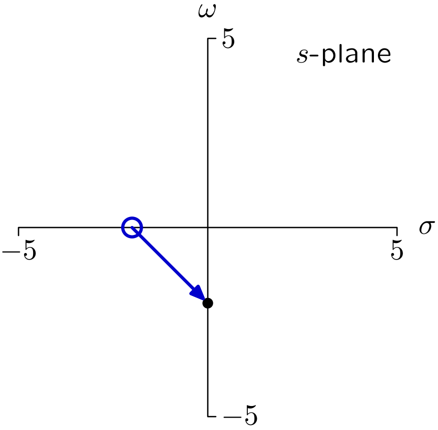

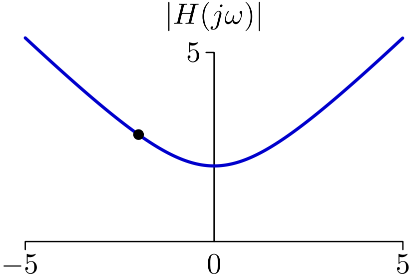

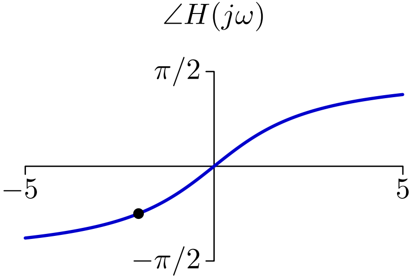

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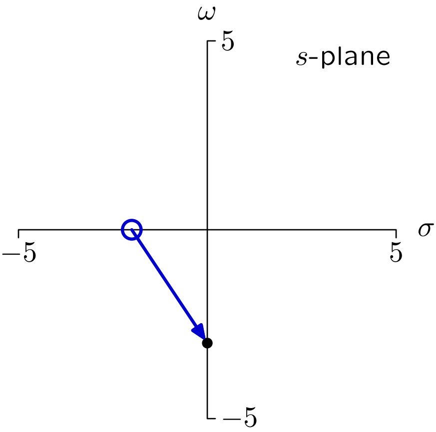

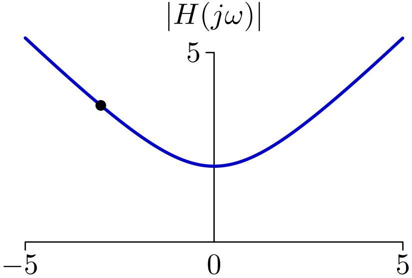

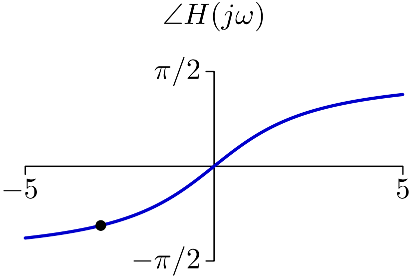

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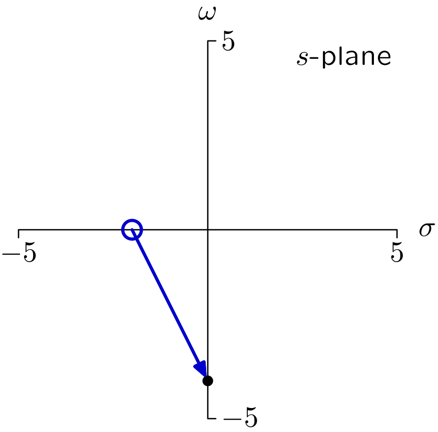

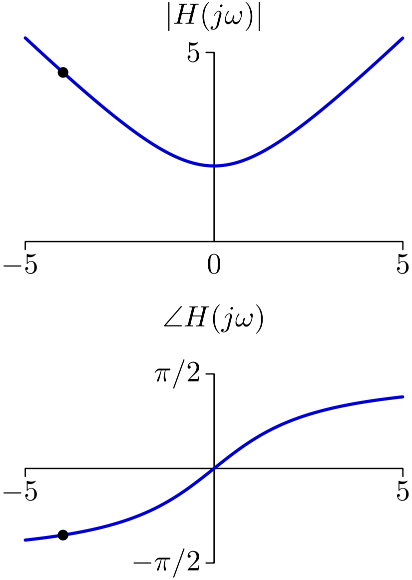

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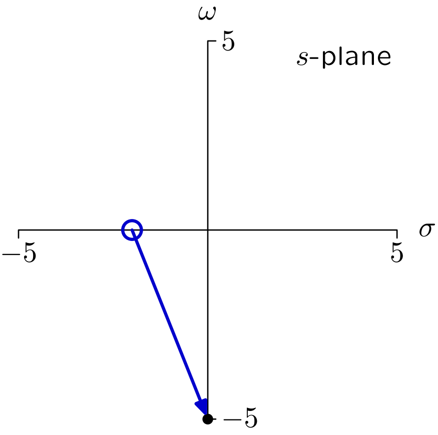
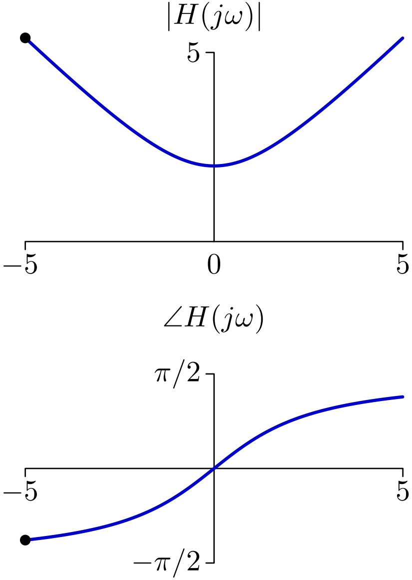
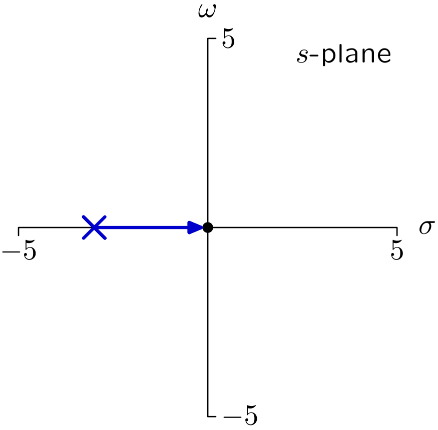
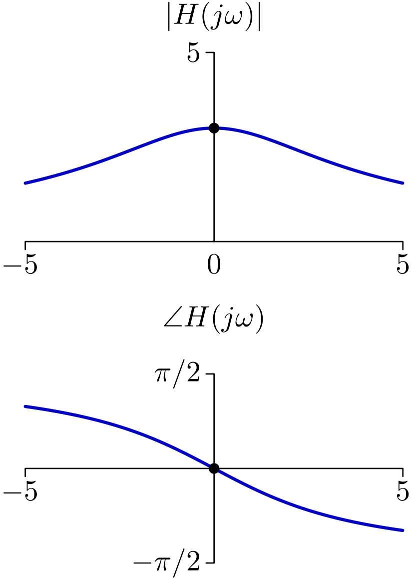




Vector Diagrams

$$H(s) = \frac{9}{s - p_1}$$





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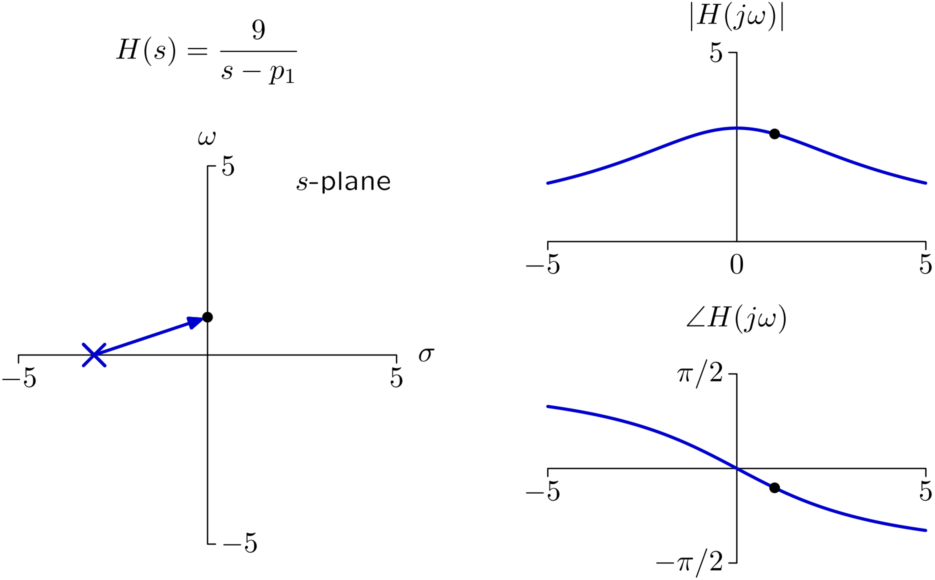

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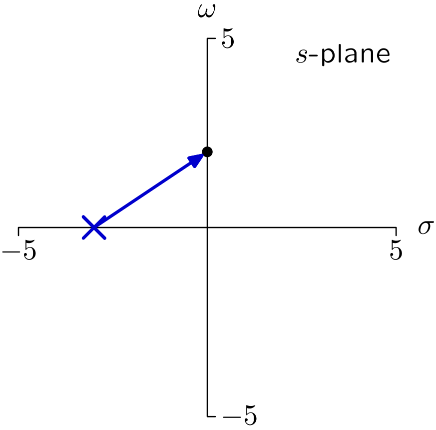

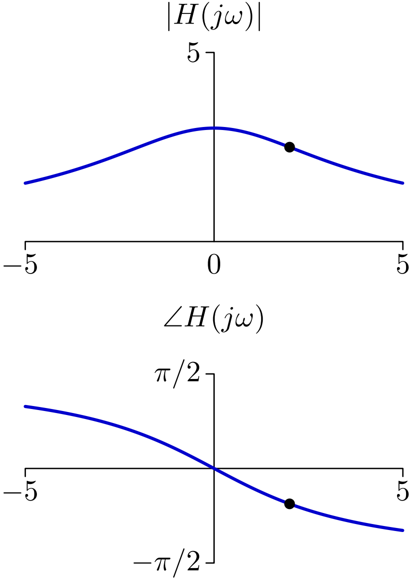

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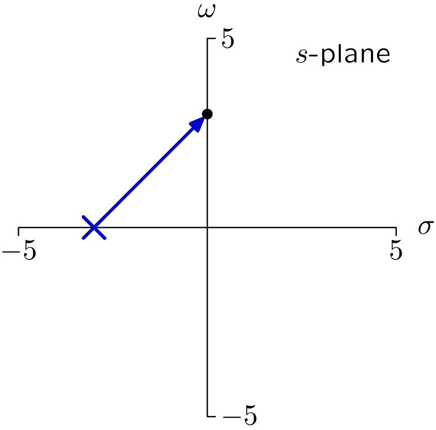

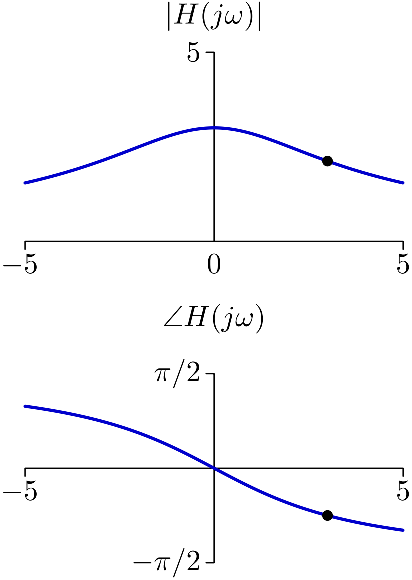

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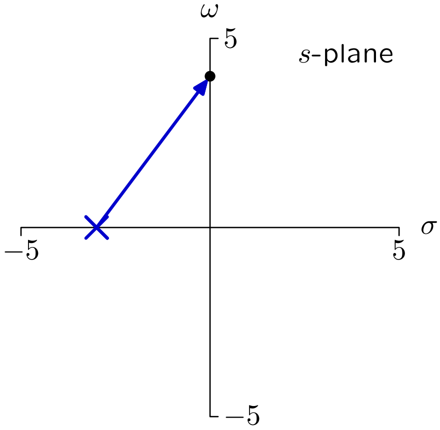

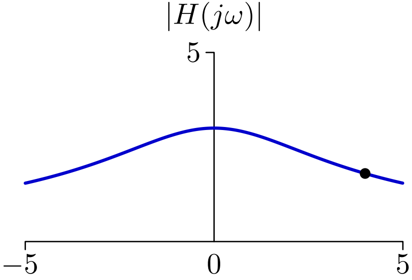

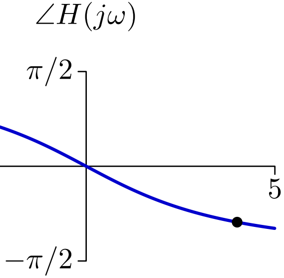

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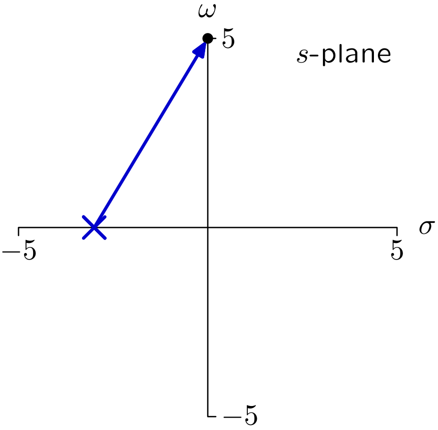
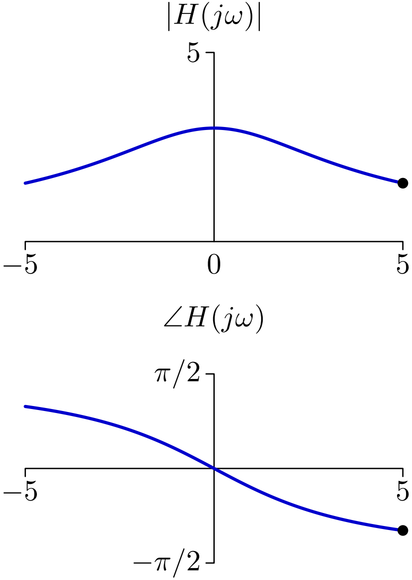
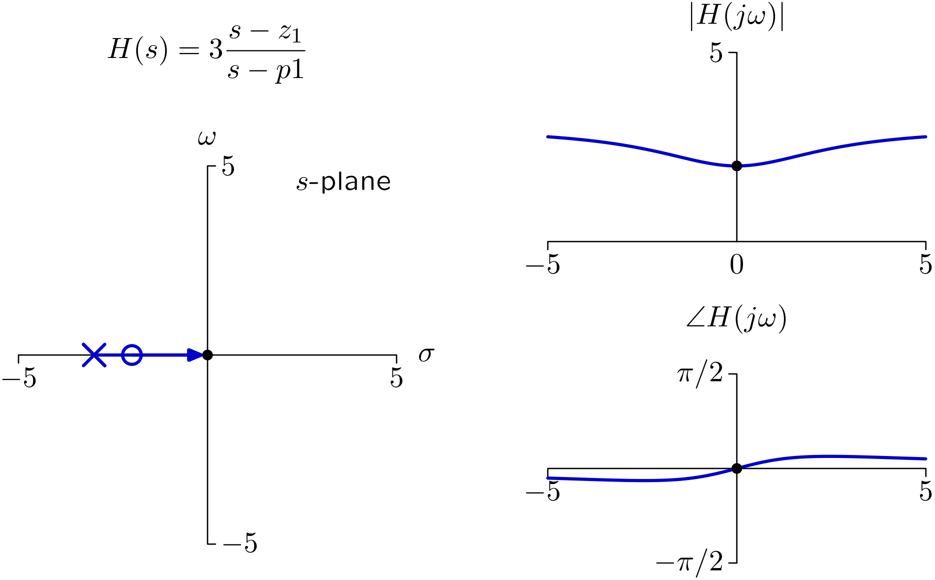




Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p1}$$



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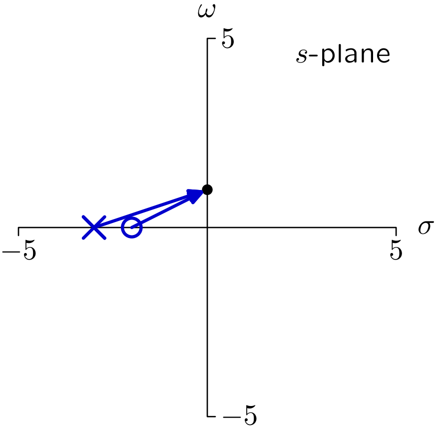

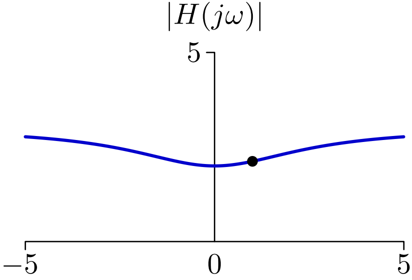

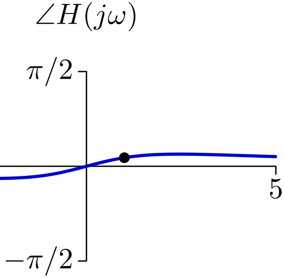

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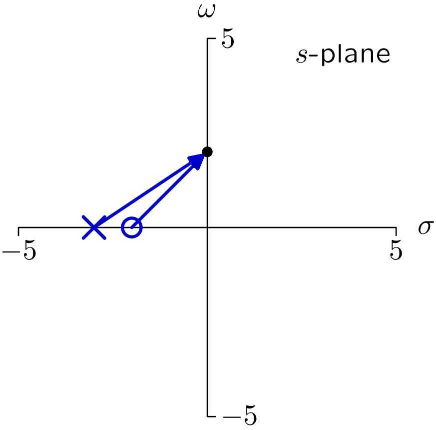
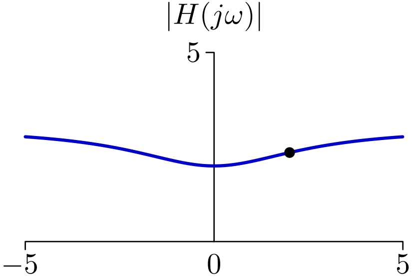
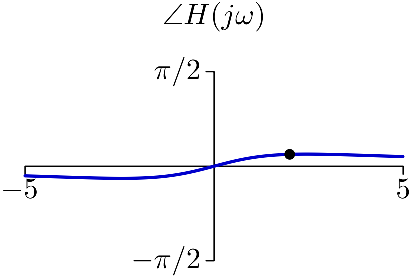
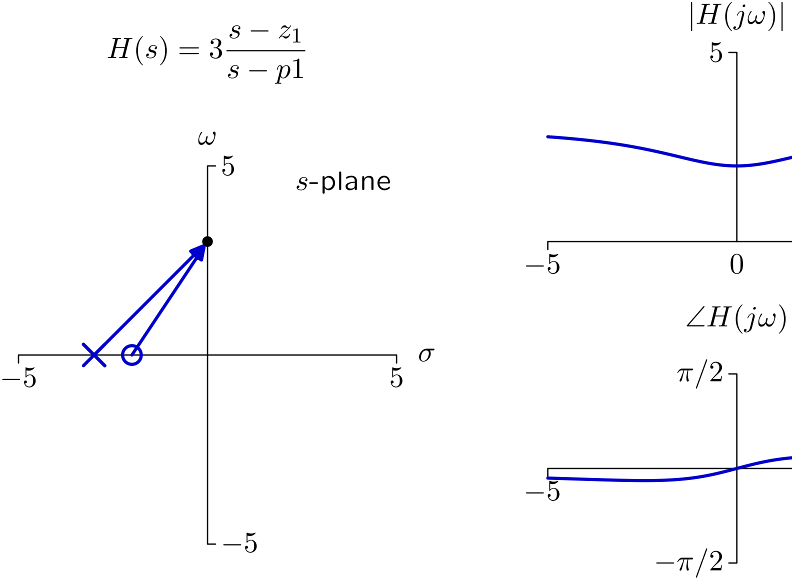






Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p1}$$



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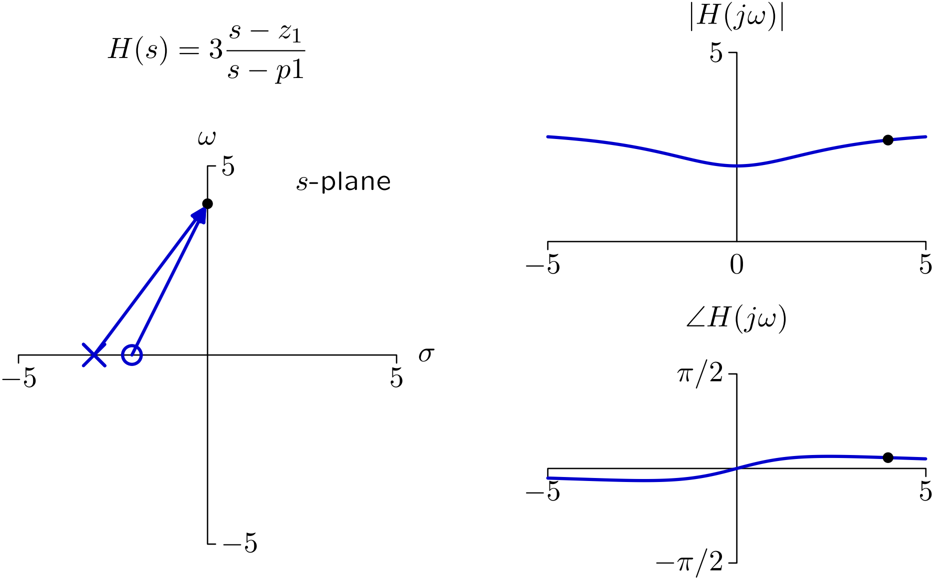

Vector Diagrams

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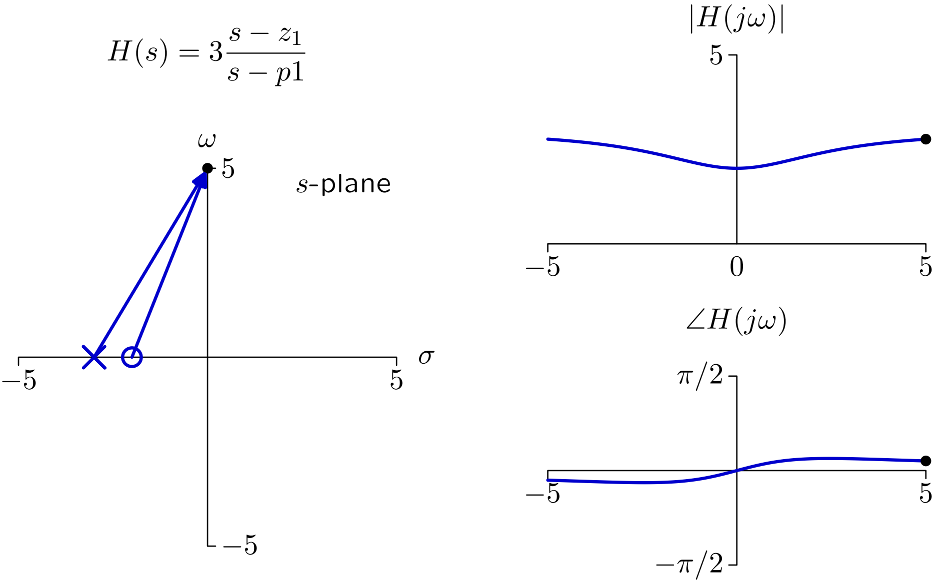
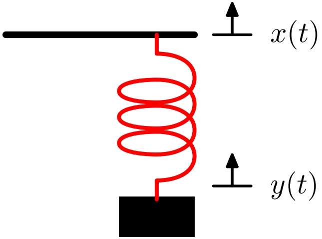


Check Yourself

Sketch the magnitude and angle of the frequency response of the mass, spring, and dashpot system.



$$F = Ma = M\ddot{y}(t) = K(x(t) - y(t)) - B\dot{y}(t)$$

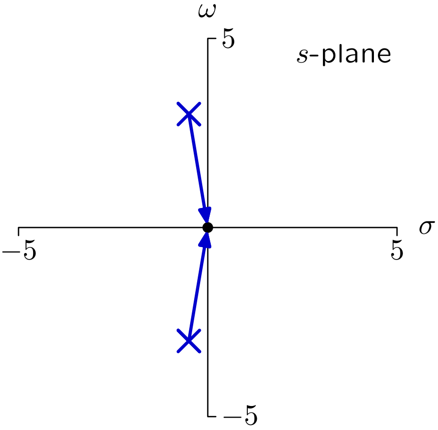
$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = Kx(t)$$

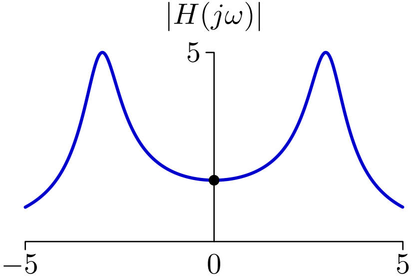
$$\left(s^2M + sB + K\right)\, Y(s) = KX(s)$$

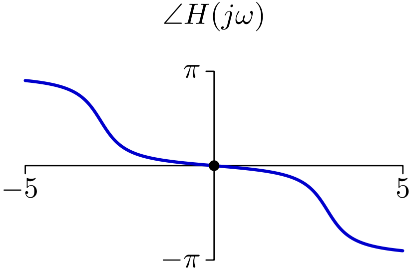
$$H(s) = \frac{K}{s^2M + sB + K}$$

Vector Diagrams

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$







Vector Diagrams

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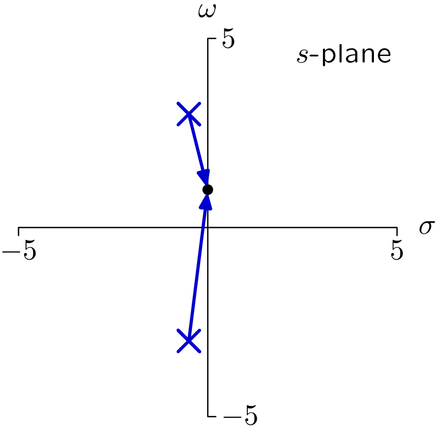

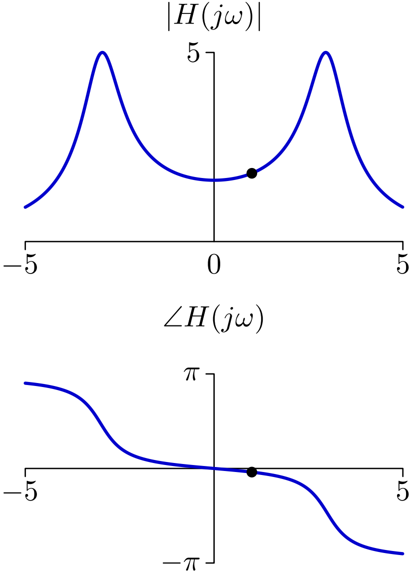

Vector Diagrams

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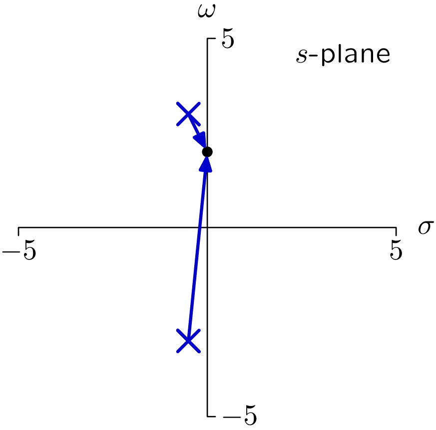

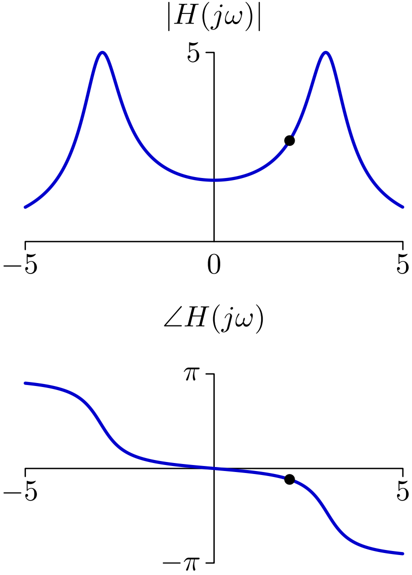

Vector Diagrams

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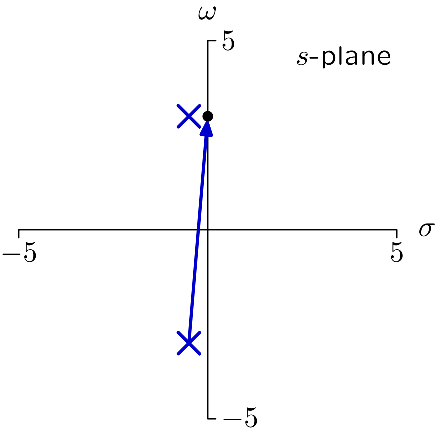

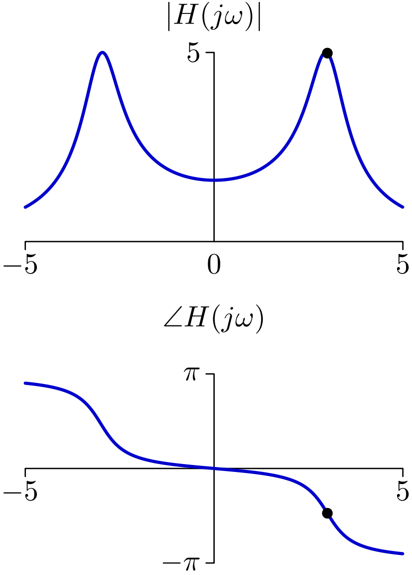

Vector Diagrams

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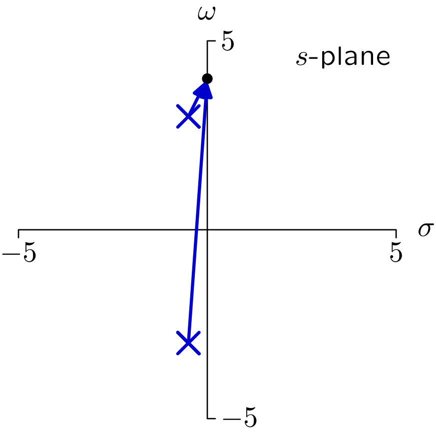

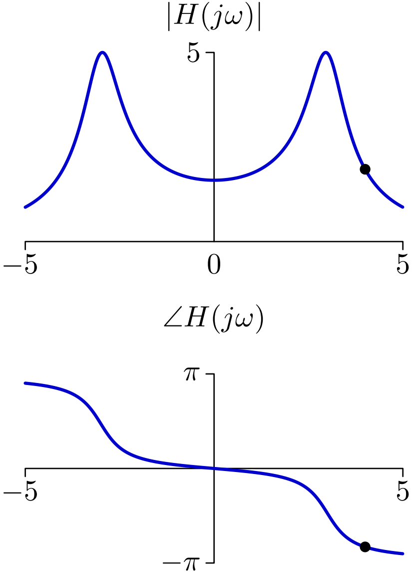

Vector Diagrams

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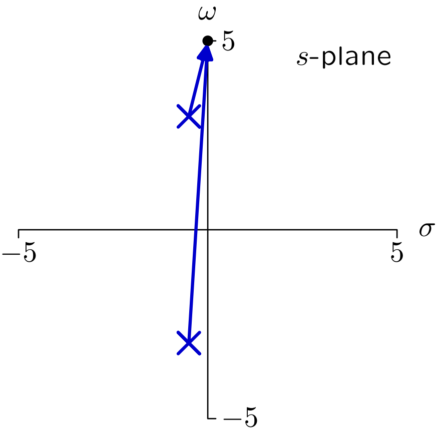
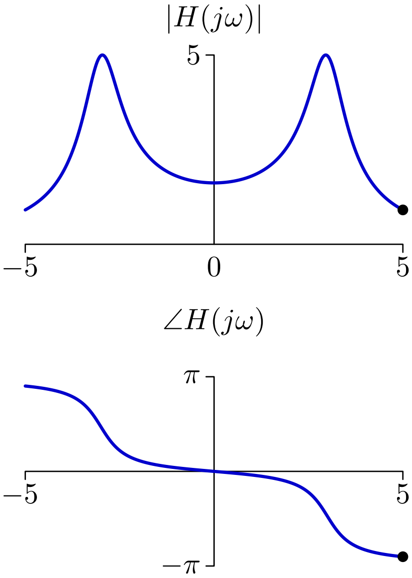




System Functions

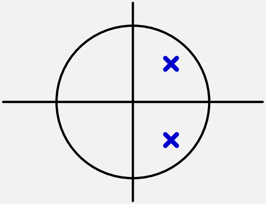
difference equation	differential equation
$y_1[n] = x_1[n] + y_1[n-1] + y_1[n-2]$	$\frac{d^2y_2(t)}{dt^2} = x_2(t) + \frac{dy_2(t)}{dt} + y_2(t)$
system function	**system function**
$H_1(z) = \frac{z^2}{z^2 - z - 1}$	$H_2(s) = \frac{1}{s^2 - s - 1}$
poles	**poles**
$z_1, z_2 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1}$	$s_1, s_2 = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1}$
homogeneous solutions	**homogeneous solutions**
$C_1 z_1^n + C_2 z_2^n$	$C_1 e^{s_1 t} + C_2 e^{s_2 t}$

- **poles** are the roots of the denominator of the system function
- each pole corresponds to a **natural frequency**
- homogeneous solution is a sum of contributions from each pole

Check Yourself

Compare two systems that each have poles as $\frac{1+j}{2}$ and $\frac{1-j}{2}$:

$$H(z) = \frac{1}{z^2 - z - \frac{1}{2}} \quad \text{and} \quad H(s) = \frac{1}{s^2 - s - \frac{1}{2}}$$



Which of the following (if any) are true?

1. the homogeneous solutions for both systems are oscillatory
2. both systems are stable
3. the homogeneous solutions for both systems converge to 0

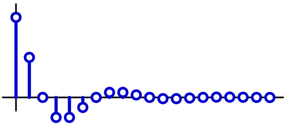
Check Yourself

Compare two systems that each have poles as $\frac{1+j}{2}$ and $\frac{1-j}{2}$:

$$H(z) = \frac{1}{z^2 - z - \frac{1}{2}} \quad \text{and} \quad H(s) = \frac{1}{s^2 - s - \frac{1}{2}}$$

The response of the discrete system (p^n) is a decaying oscillation.

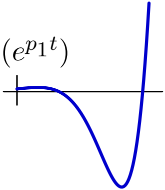
$\mathrm{Re}(p_1^n)$



n

The response of the continuous system (e^{pt} is a growing oscillation.

$\mathrm{Re}(e^{p_1 t})$



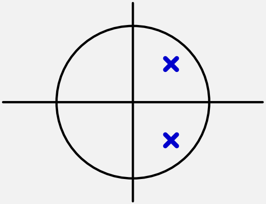
t

The responses of the discrete and continuous systems are different because the functional dependence on the pole is different.

Check Yourself

Compare two systems that each have poles as $\frac{1+j}{2}$ and $\frac{1-j}{2}$:

$$H(z) = \frac{1}{z^2 - z - \frac{1}{2}} \quad \text{and} \quad H(s) = \frac{1}{s^2 - s - \frac{1}{2}}$$



Which of the following (if any) are true? 1

1. the homogeneous solutions for both systems are oscillatory √
2. both systems are stable ✗
3. the homogeneous solutions for both systems converge to 0 ✗

Check Yourself

Today we studied the **frequency response** of a CT system.

Our most important result is that the frequency response is easily determined from the system function.

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)|\cos\left(\omega t + \angle H(j\omega)\right)$$

The frequency response is equal to the **magnitude and angle** of the system function $H(s)$ evaluated at $s = j\omega$: $H(s)\Big|_{s=j\omega}$

What is the analogous statement for a DT system?

Check Yourself

Today we studied the **frequency response** of a CT system.

Our most important result is that the frequency response is easily determined from the system function.

$$\cos(\omega t) \longrightarrow \boxed{H_{ct}(s)} \longrightarrow |H_{ct}(j\omega)|\cos\left(\omega t + \angle H_{ct}(j\omega)\right)$$

The frequency response is equal to the **magnitude and angle** of the system function $H_{ct}(s)$ evaluated at $s = j\omega$: $H_{ct}(s)\Big|_{s=j\omega}$

For DT systems

$$\cos(\Omega n) \longrightarrow \boxed{H_{dt}(z)} \longrightarrow |H_{dt}(e^{j\Omega})|\cos\left(\Omega n + \angle H_{dt}(e^{j\Omega})\right)$$

The frequency response is equal to the **magnitude and angle** of the system function $H_{dt}(s)$ evaluated at $z = e^{j\Omega}$: $H_{dt}(z)\Big|_{z=e^{j\Omega}}$

Check Yourself

eigenfunctions	z^n	e^{st}
mode associated with pole	$\left(\frac{1}{2}+j\frac{1}{2}\right)^n$	$e^{\left(\frac{1}{2}+j\frac{1}{2}\right)t}$
magnitude and angle	$\left(\frac{\sqrt{2}}{2}\right)^n e^{j\pi n/4}$	$e^{\frac{1}{2}t}e^{j\frac{1}{2}t}$

Summary

Today we developed the idea of a frequency response as an alternative way to describe the behavior of a system.

Next week we will see that the frequency response is a natural way to describe many systems and disturbances.
Frequency responses will also provide a new way to think about the design of control systems.