6.3100: Dynamic System Modeling and Control Design

Systems, Subsystems, and Basic Building Blocks

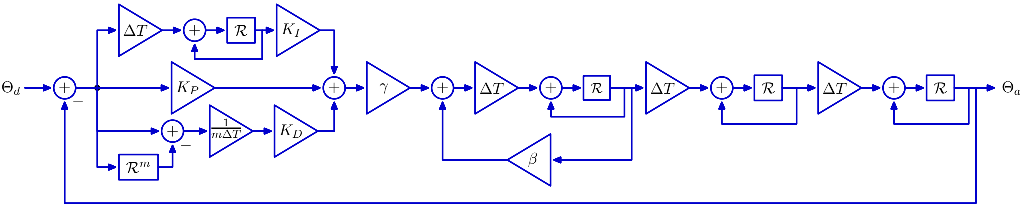
Today: Reasoning with transfer (system) functions

Wednesday: Reasoning with frequency responses

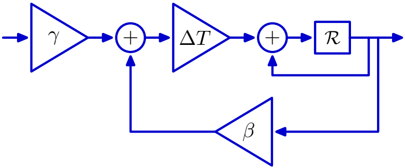
March 13, 2023

Modularity

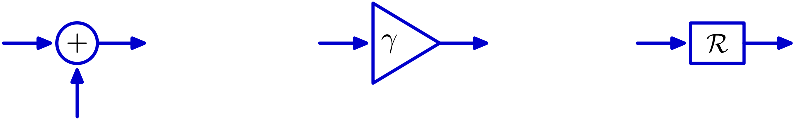
Systems



Modules

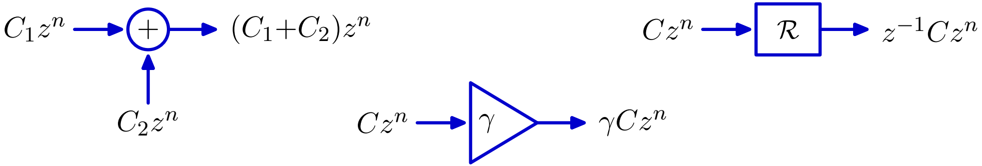


Basic Building Blocks



Similarity Across Scale

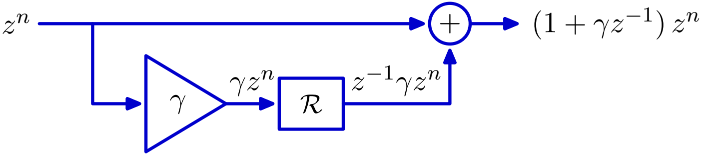
The building blocks share the **eigenfunction** property: if their input(s) have the form Cz^n then their output is a scalar multiple of their input(s).



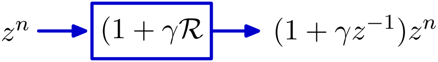
It follows that combinations of these building blocks retain the eigenfunction property.

Similarity Across Scale

Feedforward combinations have the eigenfunction property.

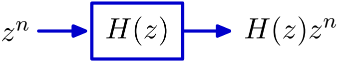


The building blocks can be combined into a single composite block:



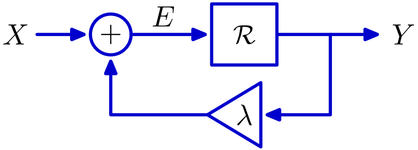
The composite block acts as a gain.

It multiplies its input by a constant ($H(z) = 1+\gamma z^{-1}$) that depends on z.



Similarity Across Scale

Feedback combinations also have the eigenfunction property.



But it takes a bit of math to show.

$$E = X + \lambda Y$$

$$Y = \mathcal{R}E = \mathcal{R}X + \lambda\mathcal{R}Y$$

$$(1 - \lambda\mathcal{R})Y = \mathcal{R}X$$

$$\frac{Y}{X} = \frac{\mathcal{R}}{1 - \lambda\mathcal{R}}$$

$$H(z) = \frac{\frac{1}{z}}{1 - \lambda\frac{1}{z}} = \frac{1}{z - \lambda}$$

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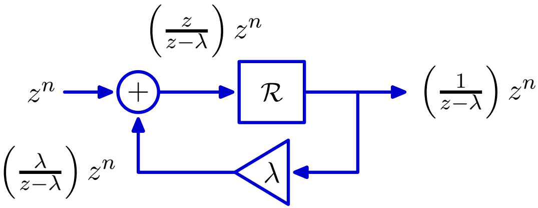

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Similarity Across Scale

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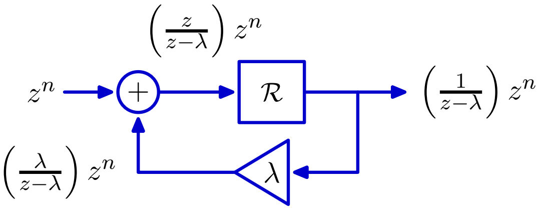

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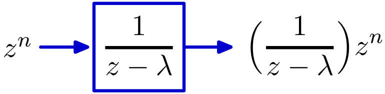
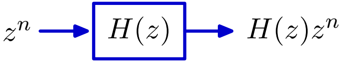


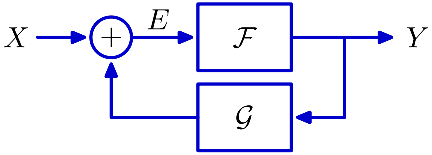
The composite block acts as a gain.

It multiplies its input by a constant $\left(H(z) = \frac{1}{z-\lambda}\right)$ that depends on z.



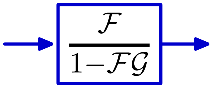
Black's Equation

Black's equation simplifies thinking about modules with feedback. Let $\mathcal{F}$ and $\mathcal{G}$ represent the **forward** and **backward** path in a feedback system.



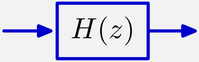
$$\left.\begin{aligned} E &= X + \mathcal{G}Y \\ Y = \mathcal{F}E &= \mathcal{F}\left(X + \mathcal{G}Y\right) \end{aligned}\right\} \left(1 - \mathcal{F}\mathcal{G}\right)Y = \mathcal{F}X$$

$$\frac{Y}{X} = \frac{\mathcal{F}}{1 - \mathcal{F}\mathcal{G}}$$

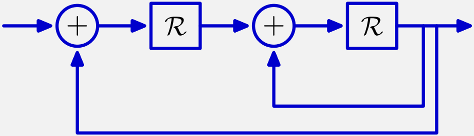


Check Yourself

Determine a representation of the following form



for the module shown below.

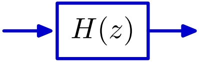


Which (if any) of the following expressions for $H(z)$ is correct?

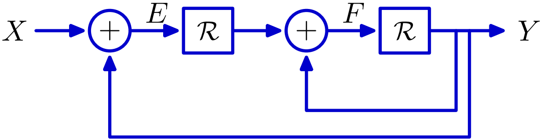
1. $\dfrac{1}{z^2+z+1}$
2. $\dfrac{1}{z^2-z-1}$
3. $\dfrac{1}{z^2+z-1}$
4. $\dfrac{1}{z^2-z+1}$
5. none of the above

Check Yourself

Determine a representation of the following form



for the module shown below.



$$Y = \mathcal{R}F$$

$$F = \mathcal{R}E + Y$$

$$E = X + Y$$

$$F = \mathcal{R}X + \mathcal{R}Y + Y$$

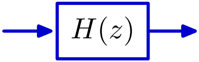
$$Y = \mathcal{R}^2 X + \mathcal{R}^2 Y + \mathcal{R}Y$$

$$\frac{Y}{X} = \frac{\mathcal{R}^2}{1 - \mathcal{R} - \mathcal{R}^2}$$

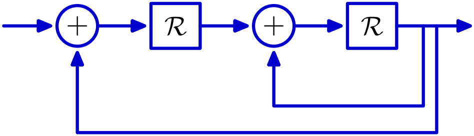
$$H(z) = \frac{1}{z^2 - z - 1}$$

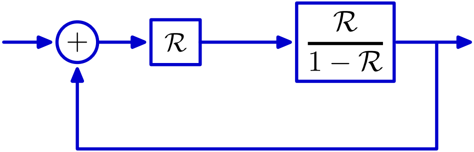
Check Yourself

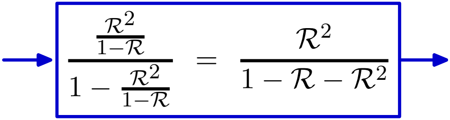
Determine a representation of the following form



for the module shown below.



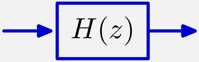




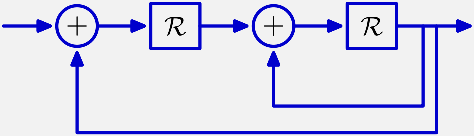
$$\frac{\frac{\mathcal{R}^2}{1-\mathcal{R}}}{1-\frac{\mathcal{R}^2}{1-\mathcal{R}}} = \frac{\mathcal{R}^2}{1-\mathcal{R}-\mathcal{R}^2}$$

Check Yourself

Determine a representation of the following form



for the module shown below.

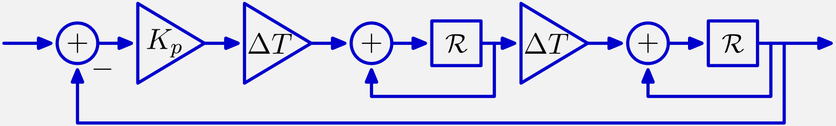


Which (if any) of the following expressions for $H(z)$ is correct? 2

1. $\dfrac{1}{z^2+z+1}$
2. $\dfrac{1}{z^2-z-1}$
3. $\dfrac{1}{z^2+z-1}$
4. $\dfrac{1}{z^2-z+1}$
5. none of the above

Check Yourself

Consider the following system.

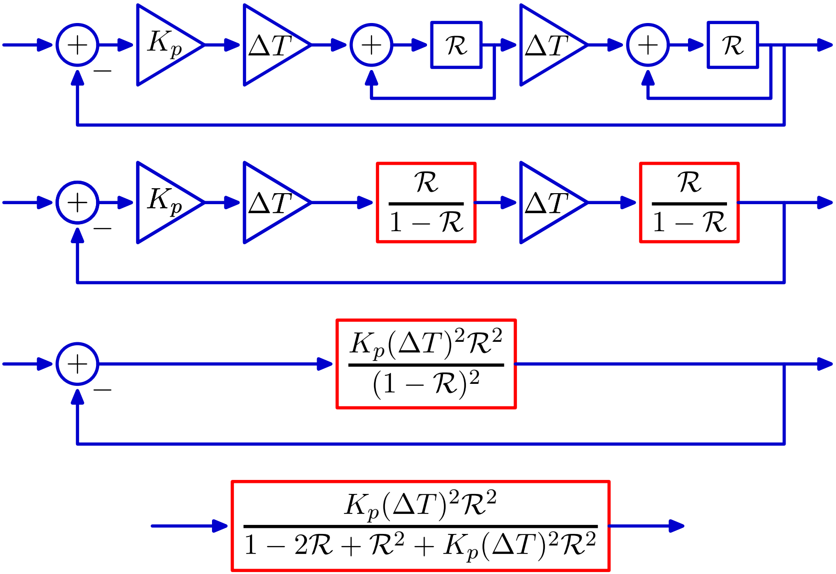


What are the poles of this system?

1. $j\Delta T\sqrt{K_p}$
2. $-1 \pm j\Delta T\sqrt{K_p}$
3. $1 \pm j\Delta T\sqrt{K_p}$
4. $\pm\Delta T\sqrt{K_p}$
5. none of the above

Check Yourself

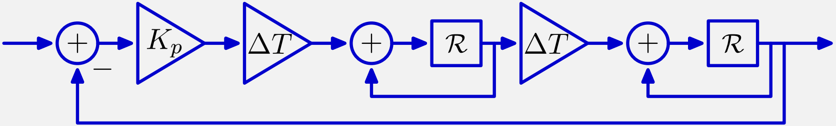
Use Black's equation to replace feedback loops with system functionals.



poles at $z = 1 \pm j\Delta T\sqrt{K_p}$

Check Yourself

Consider the following system.

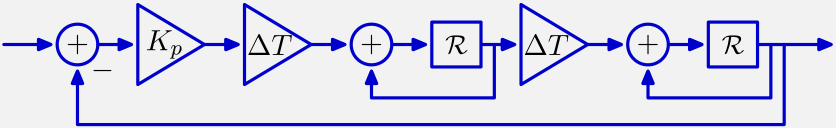


What are the poles of this system? 3

1. $j\Delta T\sqrt{K_p}$
2. $-1 \pm j\Delta T\sqrt{K_p}$
3. $1 \pm j\Delta T\sqrt{K_p}$
4. $\pm\Delta T\sqrt{K_p}$
5. none of the above

Check Yourself

Determine a continuous-time approximation to the following discrete-time system.



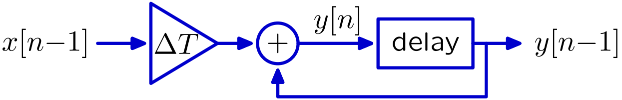
Determine a block diagram for a continuous-time approximation to this system.

What are the poles of the continuous-time approximation?

Compare the poles of the CT approximation to those of the original DT system.

Continuous-Time Approximation of a Discrete-Time System

Consider one section of the following form:

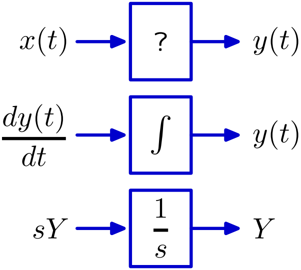


$$y[n] = y[n-1] + \Delta T x[n-1]$$

$$\frac{y[n] - y[n-1]}{\Delta T} = x[n-1]$$

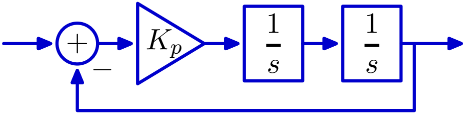
Take the limit as $\Delta T \to 0$:

$$\frac{dy(t)}{dt} = x(t)$$



Continuous-Time Approximation of a Discrete-Time System

Substitute the continuous time integrator for the discrete-time sum.



Solve for the closed-loop system function $H = \frac{Y}{X}$ using Black's equation.

$$H(s) = \frac{\frac{K_p}{s^2}}{1 + \frac{K_p}{s^2}} = \frac{K_p}{s^2 + K_p}$$

The poles are at $s = \pm j\sqrt{K_p}$.

Continuous-Time Approximation of a Discrete-Time System

Compare the poles of the DT system to those of the CT approximation.

DT: $z_p = 1 \pm j\Delta T\sqrt{K_p}$.

CT: $s_p = \pm j\sqrt{K_p}$.

Continuous-Time Approximation of a Discrete-Time System

Compare the poles of the DT system to those of the CT approximation.

DT: $z_p = 1 \pm j\Delta T\sqrt{K_p}$.

CT: $s_p = \pm j\sqrt{K_p}$.

Compare fundamental modes.

DT: $(1 \pm j\Delta T\sqrt{K_p})^n$

CT: $e^{\pm j\sqrt{K_p}t}$

Compare samples of CT and DT modes.

$$e^{\pm j\sqrt{K_p}t} = e^{\pm j\sqrt{K_p}\Delta T n} = \left(e^{\pm j\sqrt{K_p}\Delta T}\right)^n$$

Expand exponential in Maclaurin series.

$$e^{j\theta} = 1 + j\theta - \frac{1}{2!}\theta^2 - j\frac{1}{3!}\theta^3 + \cdots \approx \quad 1 + j\theta \quad \text{if} \quad \theta << 1$$

$$e^{\pm j\sqrt{K_p}t} = \left(e^{\pm j\sqrt{K_p}\Delta T}\right)^n \approx \left(1 \pm j\sqrt{K_p}\Delta T\right)^n \quad \text{if} \quad \sqrt{K_p}\Delta T << 1$$

Continuous-Time Approximation of a Discrete-Time System

Compare the poles of the DT system to those of the CT approximation.

DT: $z_p = 1 \pm j\Delta T\sqrt{K_p}$.

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Compare samples of CT and DT modes.

$$e^{\pm j\sqrt{K_p}t} = e^{\pm j\sqrt{K_p}\Delta Tn} = \left(e^{\pm j\sqrt{K_p}\Delta T}\right)^n$$

Expand exponential in Maclaurin series.

$$e^{j\theta} = 1 + j\theta - \frac{1}{2!}\theta^2 - j\frac{1}{3!}\theta^3 + \cdots \approx \quad 1+j\theta \quad \text{if} \quad \theta << 1$$

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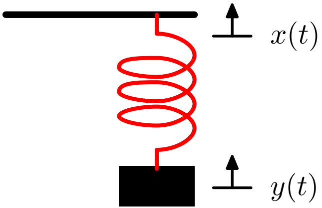
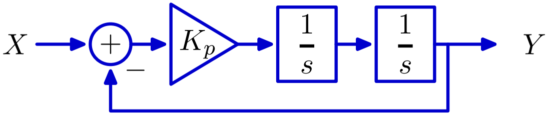
Mass and Spring System

A similar block diagram can be used to represent a mass and spring system.



$$F = K\Big(x(t) - y(t)\Big) = M\frac{d^2y(t)}{dt^2}$$



What is the value of K_p?

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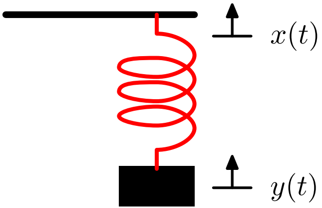

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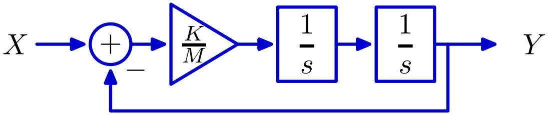

The constant $K_p = \frac{K}{M}$.

The poles are at $s = \pm j\sqrt{\frac{K}{M}}$.

Summary

Today we focused on the modular analysis of complex systems.

Black's equation is an effective tool for thinking about feedback.

Analysis of CT systems is similar to analysis of DT systems.

Interpretation of poles is different in DT and CT.