6.3100: Dynamic System Modeling and Control Design

Systems, Subsystems, and Basic Building Blocks

Modularity

Last Time: Reasoning with **transfer (system) functions**

Today: Reasoning with **frequency responses**

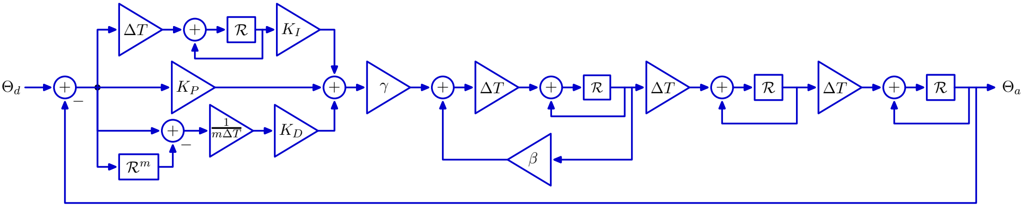
March 15, 2023

Modularity

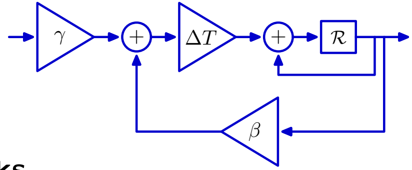
Last time: breaking complicated systems into modules.

→ Black's equation and the importance of transfer (system) functions.

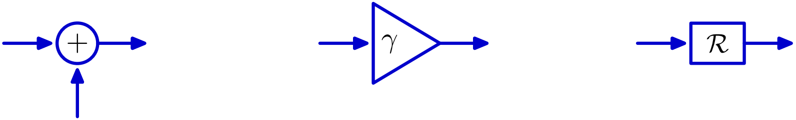
Systems



Modules



Basic Building Blocks



Modularity

Today: Modular construction of frequency responses using Bode plots.

Modularity

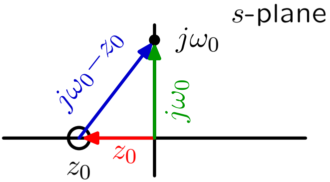
Today: Modular construction of frequency responses using Bode plots.
→ builds on the vector analysis method from last week.

The **frequency response** of a system composed of adders, gains, and integrators

$$\cos(\omega_0 t) \longrightarrow \boxed{H(j\omega_0)} \longrightarrow |H(j\omega_0)|\cos\left(\omega_0 t + \angle H(j\omega_0)\right)$$

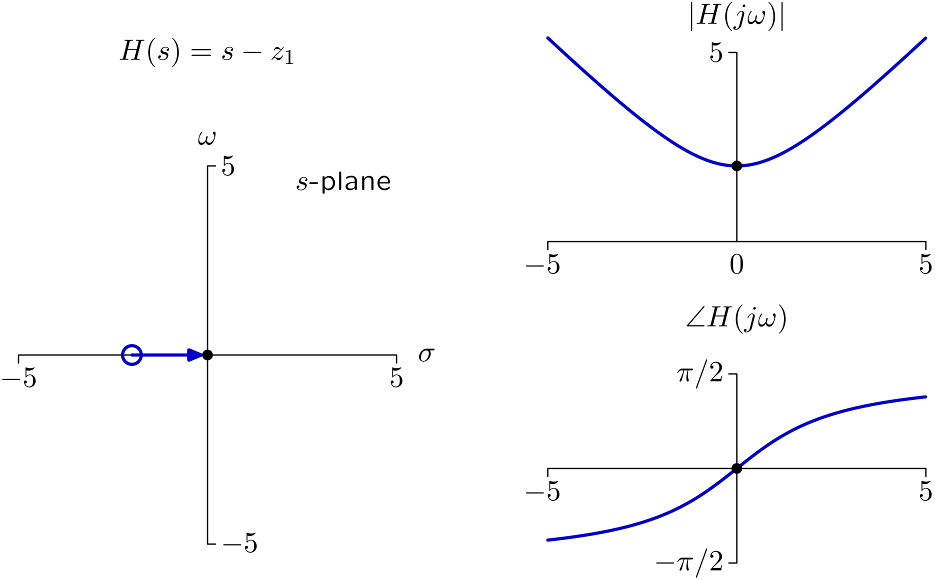
can be determined from **vectors** associated with the system's poles/zeros.

$$H(j\omega_0) = K\frac{(j\omega_0 - z_0)(j\omega_0 - z_1)(j\omega_0 - z_2)\cdots}{(j\omega_0 - p_0)(j\omega_0 - p_1)(j\omega_0 - p_2)\cdots}$$



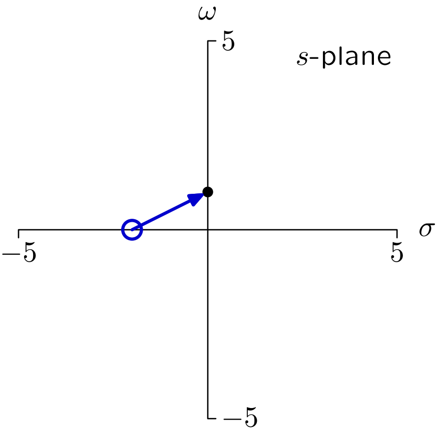
Frequency Response: Single Zero

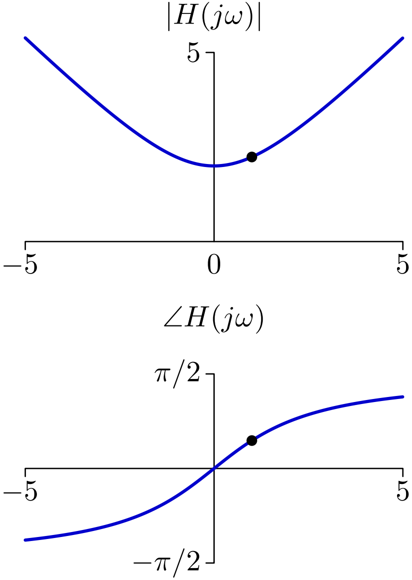
$$H(s) = s - z_1$$



Frequency Response: Single Zero

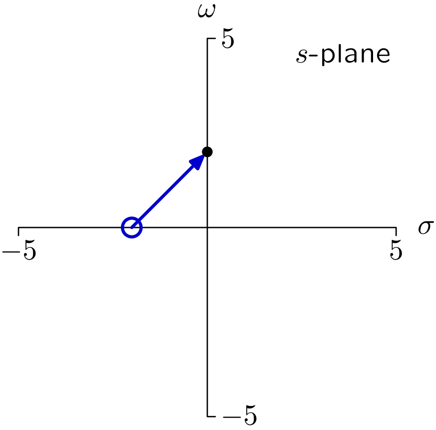
$$H(s) = s - z_1$$

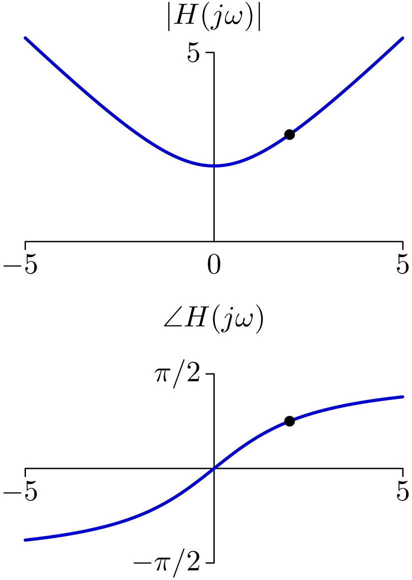




Frequency Response: Single Zero

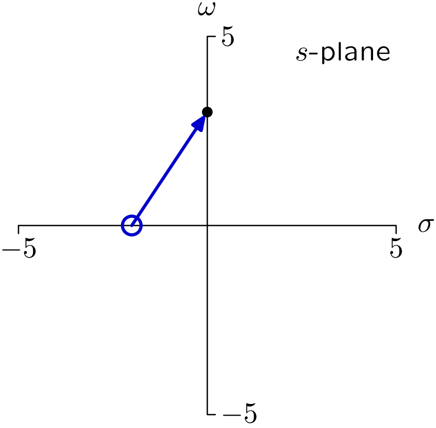
$$H(s) = s - z_1$$

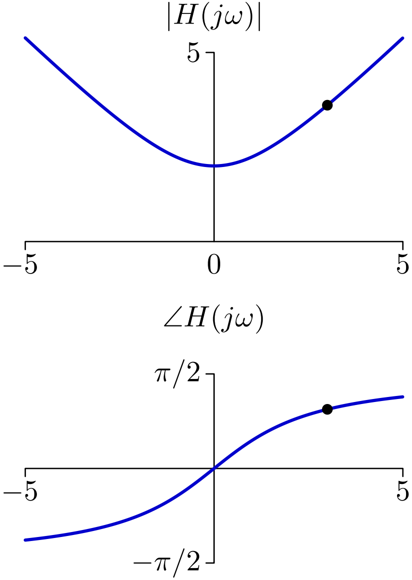




Frequency Response: Single Zero

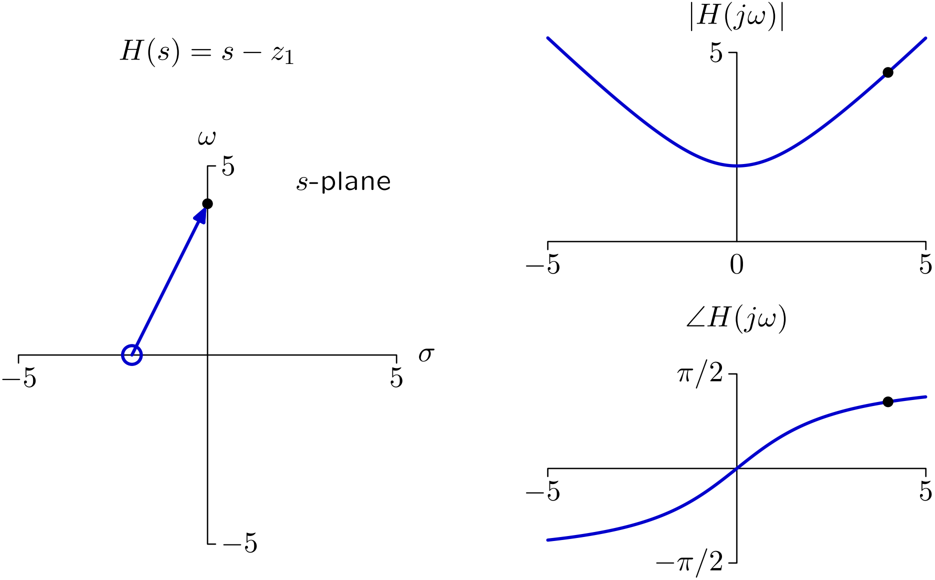
$$H(s) = s - z_1$$



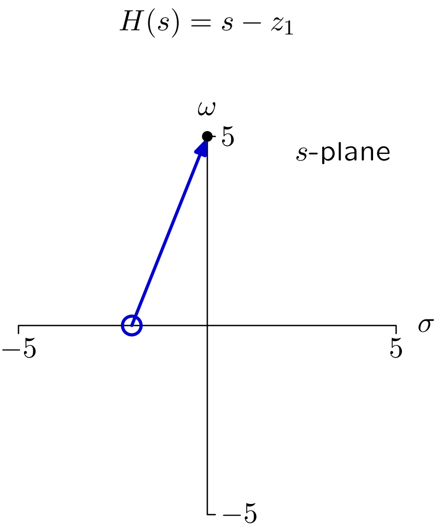


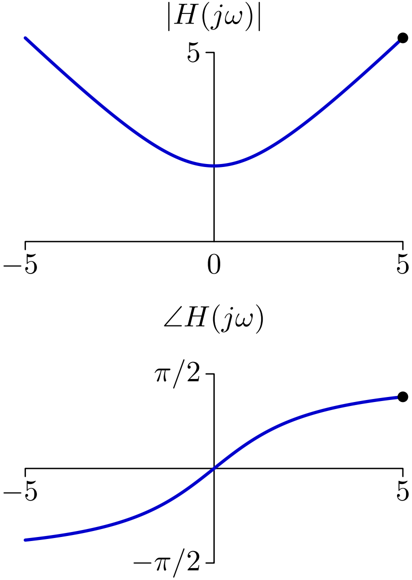
Frequency Response: Single Zero

$$H(s) = s - z_1$$



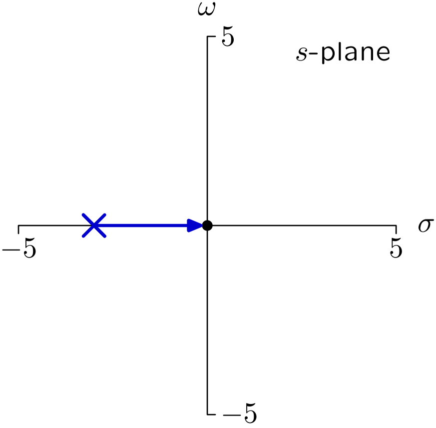
Frequency Response: Single Zero

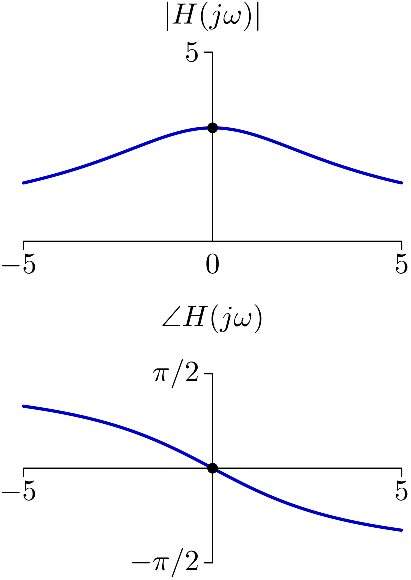




Frequency Response: Single Pole

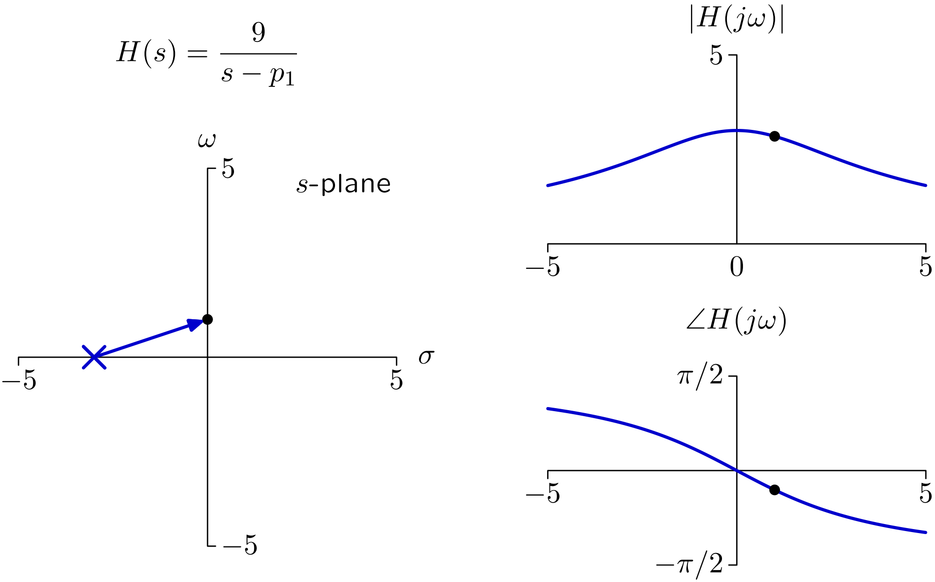
$$H(s) = \frac{9}{s - p_1}$$





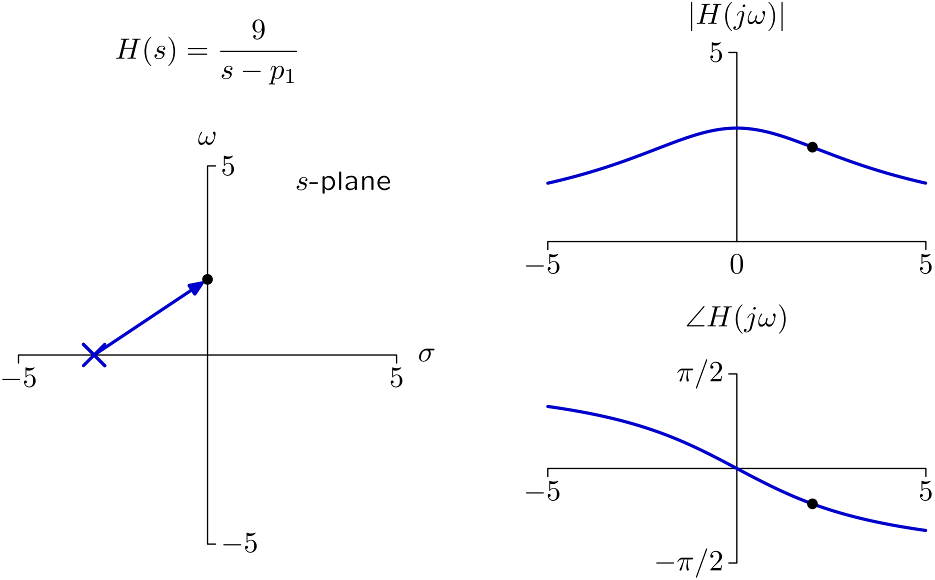
Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



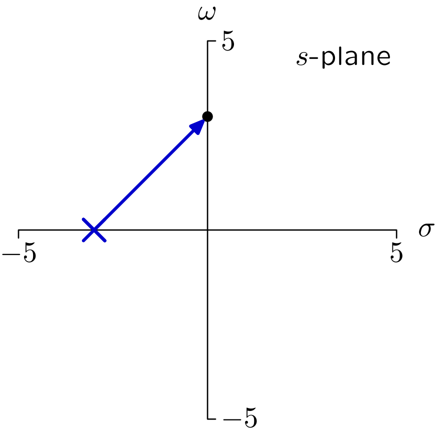
Frequency Response: Single Pole

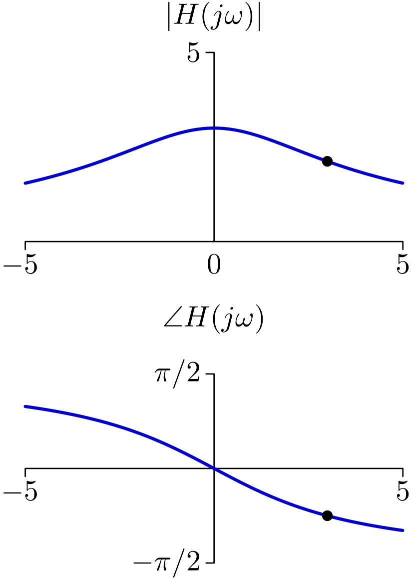
$$H(s) = \frac{9}{s - p_1}$$



Frequency Response: Single Pole

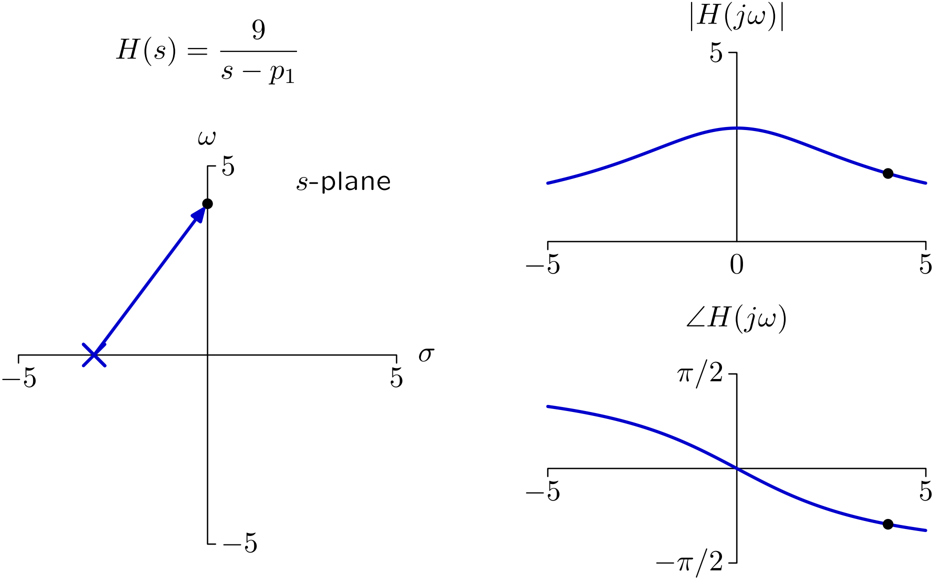
$$H(s) = \frac{9}{s - p_1}$$





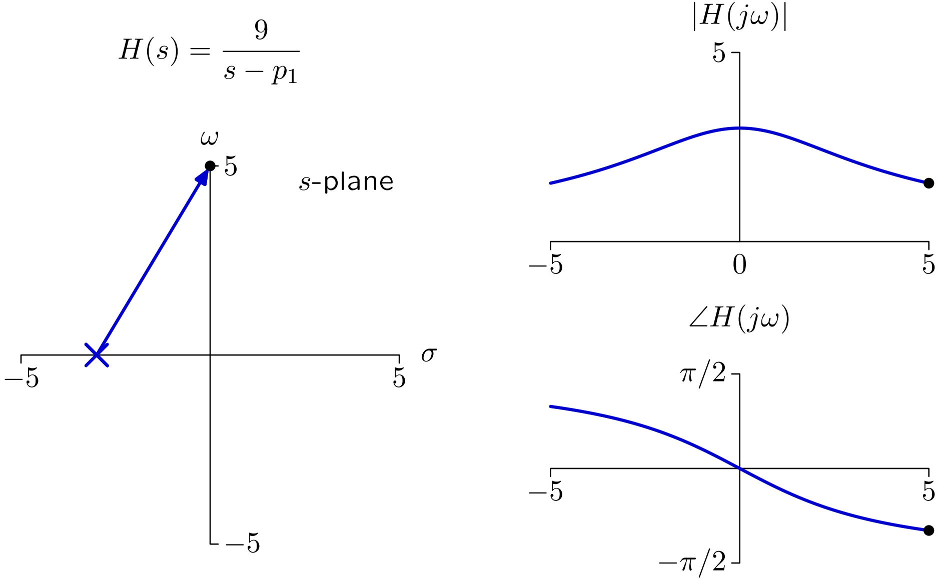
Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



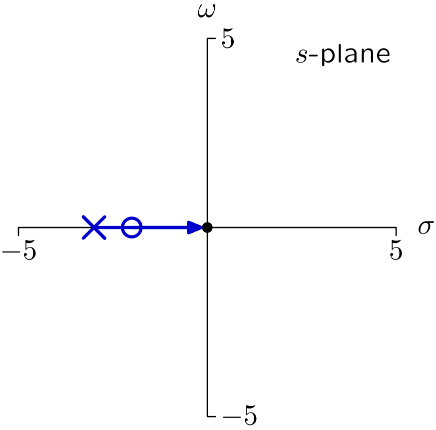
Frequency Response: Single Pole

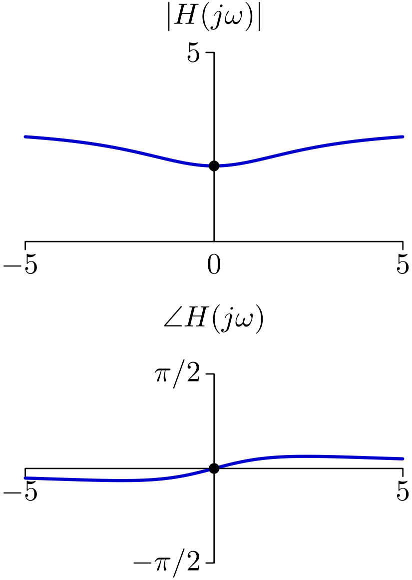
$$H(s) = \frac{9}{s - p_1}$$



Frequency Response: One Zero and One Pole

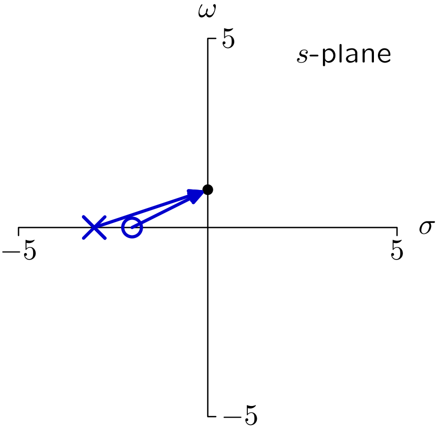
$$H(s) = 3\,\frac{s - z_1}{s - p_1}$$

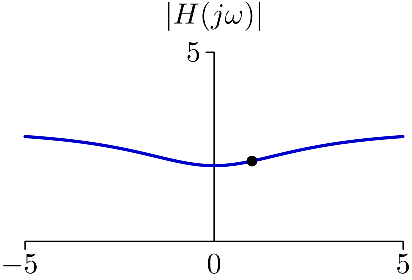


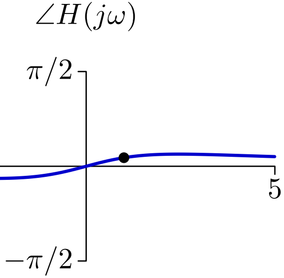


Frequency Response: One Zero and One Pole

$$H(s) = 3\,\frac{s - z_1}{s - p_1}$$

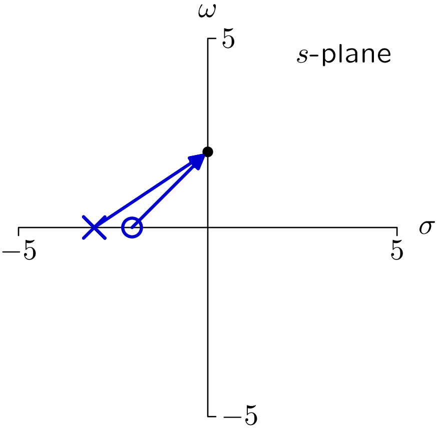


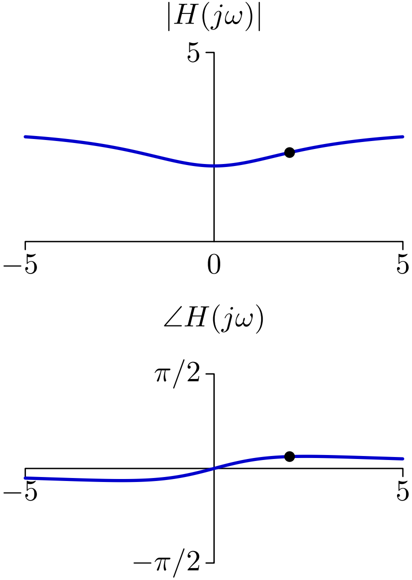




Frequency Response: One Zero and One Pole

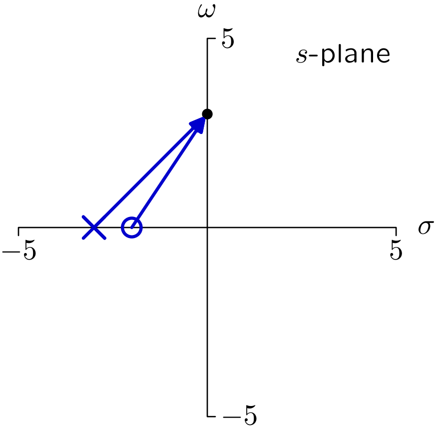
$$H(s) = 3\,\frac{s - z_1}{s - p_1}$$

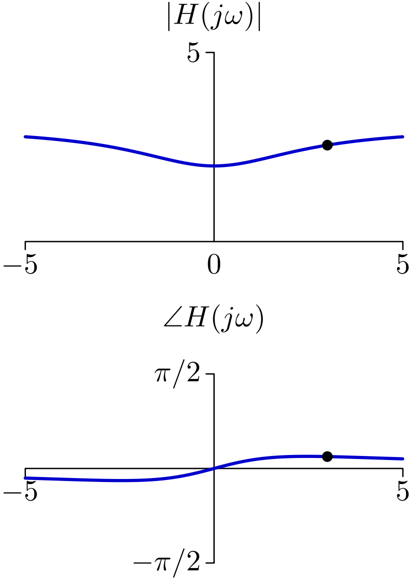




Frequency Response: One Zero and One Pole

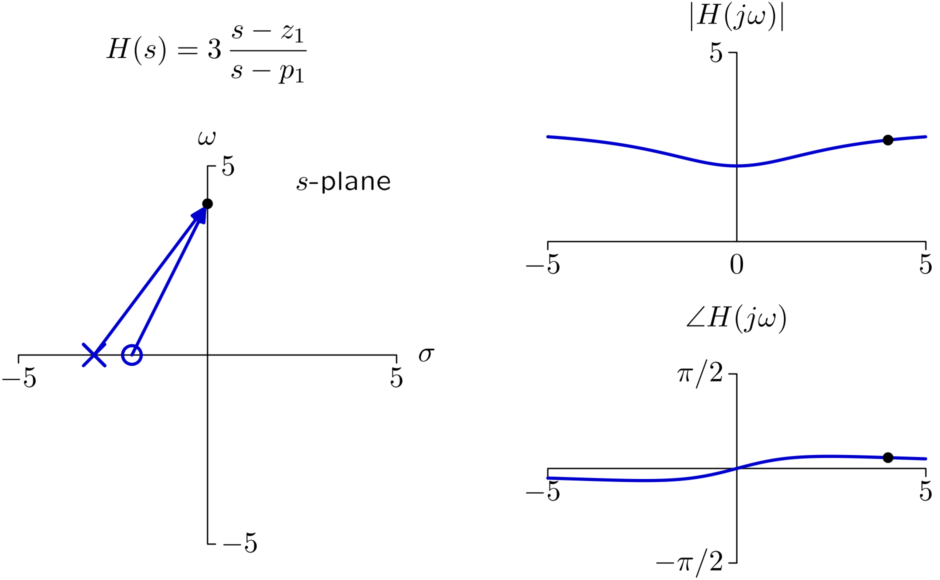
$$H(s) = 3\,\frac{s - z_1}{s - p_1}$$





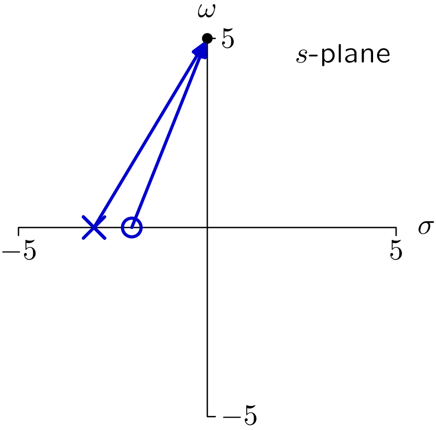
Frequency Response: One Zero and One Pole

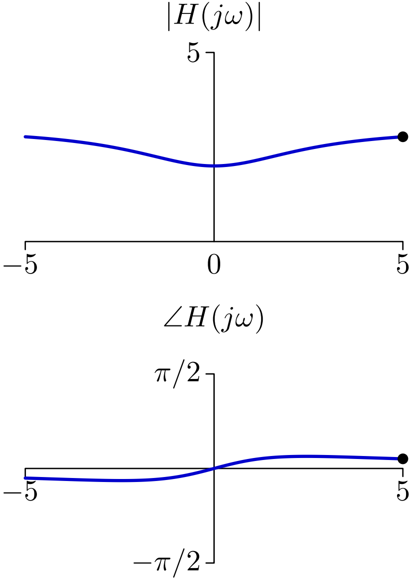
$$H(s) = 3\,\frac{s - z_1}{s - p_1}$$



Frequency Response: One Zero and One Pole

$$H(s) = 3\,\frac{s - z_1}{s - p_1}$$





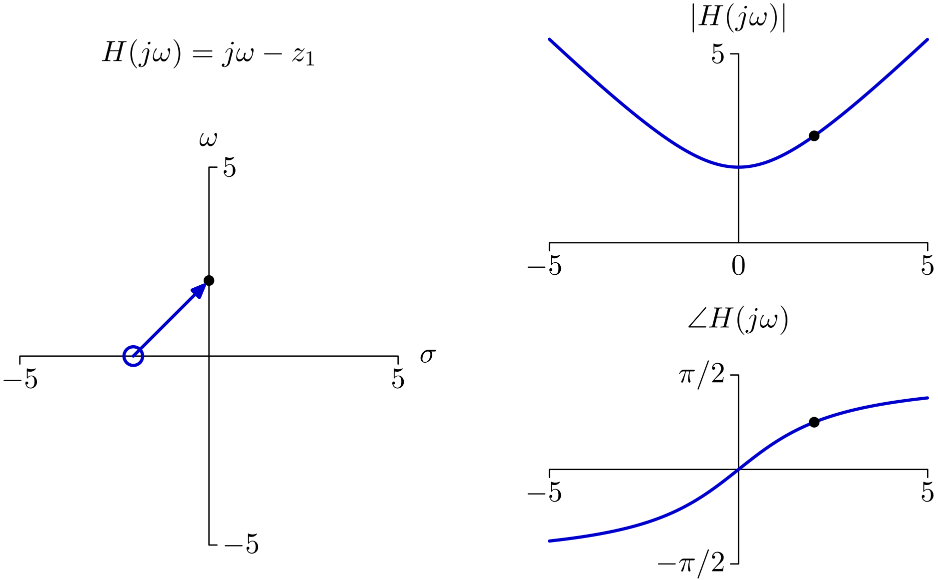
Bode Plots

Vector diagrams: simple and powerful way to characterize frequency response with just a few vectors.

Bode plots are even simpler.

Asymptotic Behavior: Single Zero

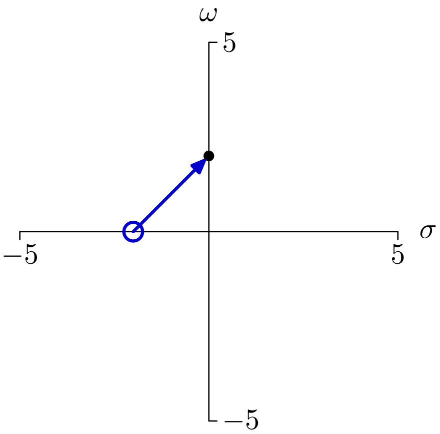
The magnitude response is simple at low and high frequencies.

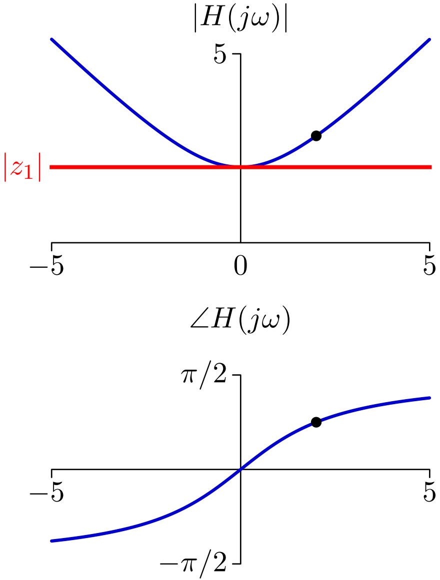


Asymptotic Behavior: Single Zero

The magnitude response is simple at low and high frequencies.

$$H(j\omega) = j\omega - z_1$$

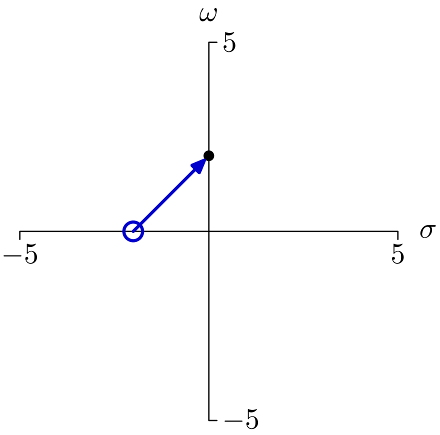


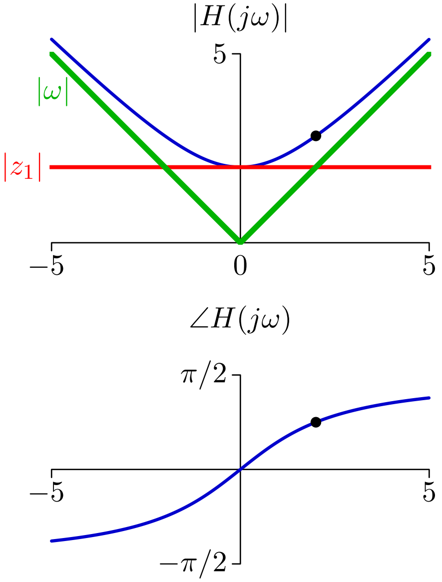


Asymptotic Behavior: Single Zero

The magnitude response is simple at low and high frequencies.

$$H(j\omega) = j\omega - z_1$$

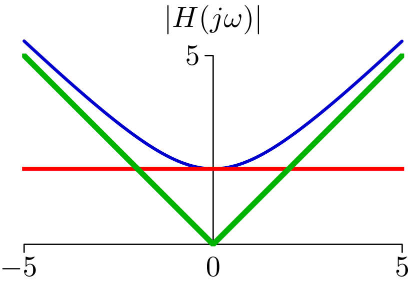


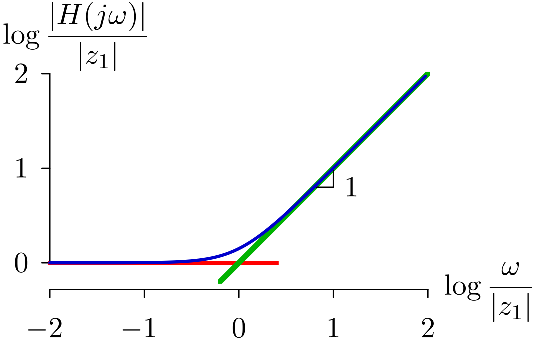


Asymptotic Behavior: Single Zero

Two asymptotes provide a good approximation on log-log axes.

$$H(s) = s - z_1$$





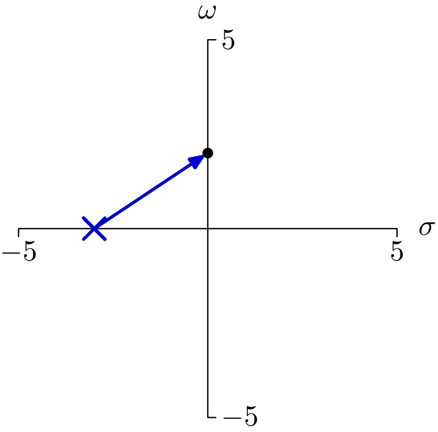
$$\lim_{\omega \to 0} |H(j\omega)| = |z_1|$$

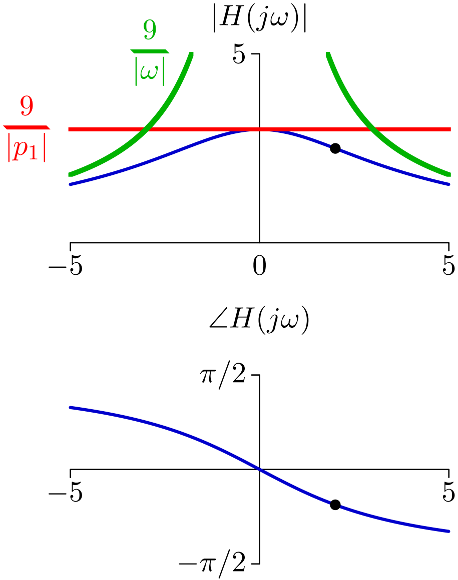
$$\lim_{\omega \to \infty} |H(j\omega)| = \omega$$

Asymptotic Behavior: Single Pole

The magnitude response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$

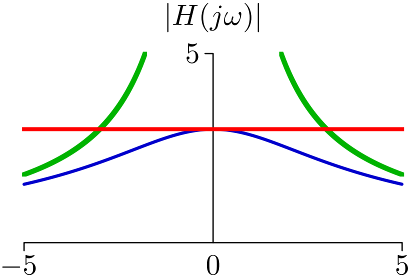


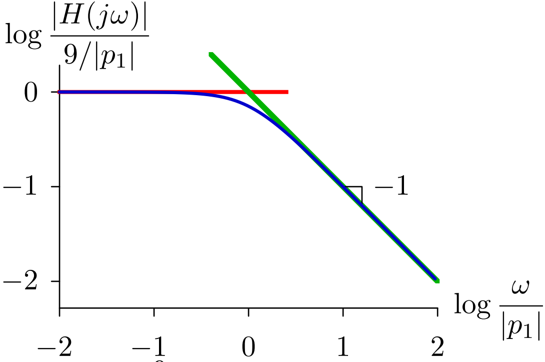


Asymptotic Behavior: Single Pole

Two asymptotes provide a good approximation on log-log axes.

$$H(s) = \frac{9}{s - p_1}$$





$$\lim_{\omega \to 0} |H(j\omega)| = \frac{9}{|p_1|}$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \frac{9}{\omega}$$

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$

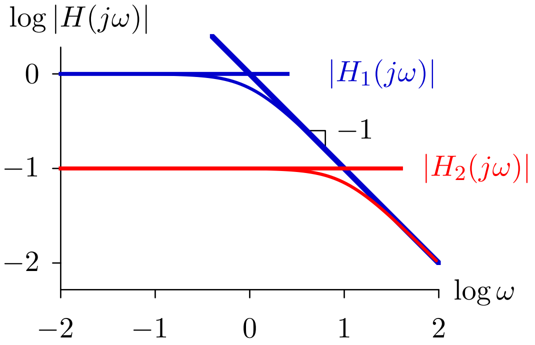
The former can be transformed into the latter by

1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \qquad \text{and} \qquad H_2(s) = \frac{1}{s+10}$$



Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{1}{s+10}$$

The former can be transformed into the latter by 3

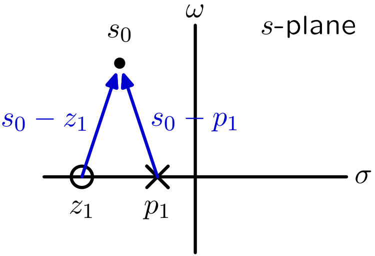
1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

no scaling in either vertical or horizontal directions!

Asymptotic Behavior of More Complicated Systems

Constructing $H(s_0)$.

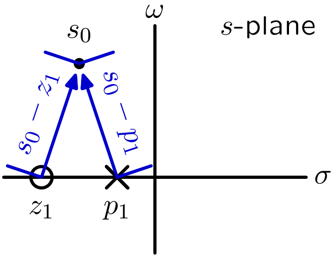
$$H(s_0) = K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \quad \begin{matrix} \leftarrow \text{ product of vectors for zeros} \\ \leftarrow \text{ product of vectors for poles} \end{matrix}$$



Asymptotic Behavior of More Complicated Systems

The magnitude of a product is the product of the magnitudes.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q}|s_0 - z_q|}{\prod_{p=1}^{P}|s_0 - p_p|}$$



Bode Plot

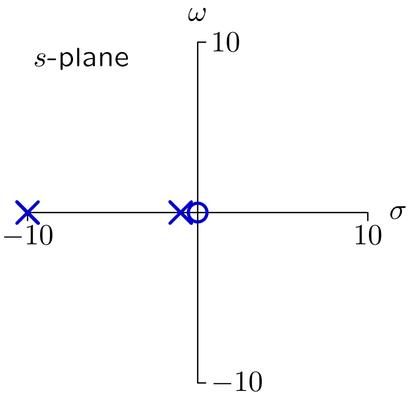
The log of the magnitude is a sum of logs.

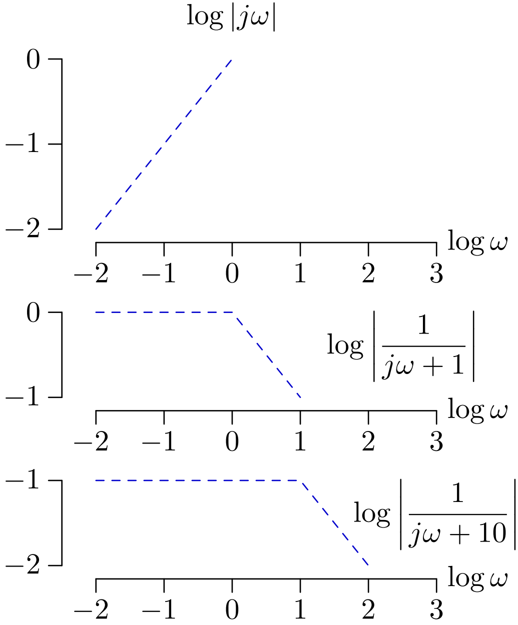
$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q}|s_0 - z_q|}{\prod_{p=1}^{P}|s_0 - p_p|}$$

$$\log|H(j\omega)| = \log|K| + \sum_{q=1}^{Q} \log|j\omega - z_q| - \sum_{p=1}^{P} \log|j\omega - p_p|$$

Bode Plot: Adding Instead of Multiplying

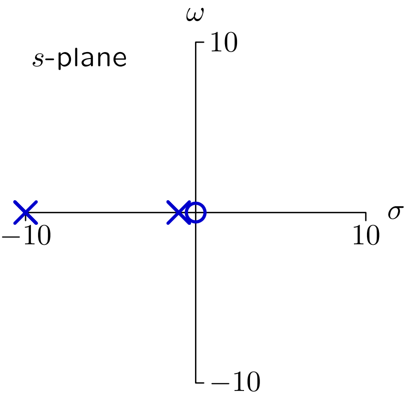
$$H(s) = \frac{s}{(s+1)(s+10)}$$

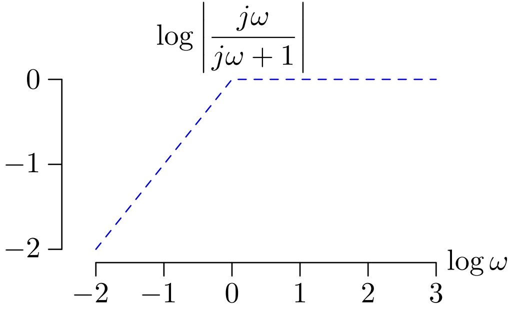


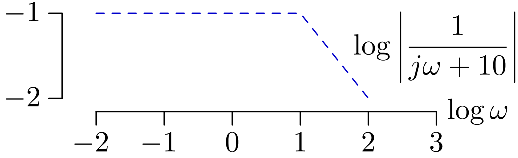


Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$

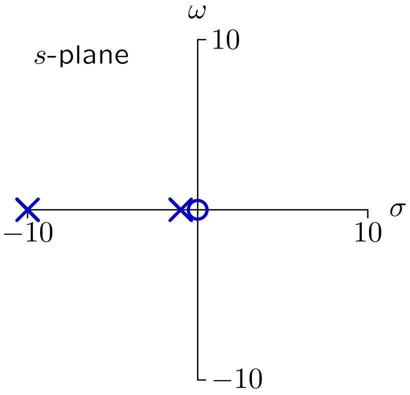


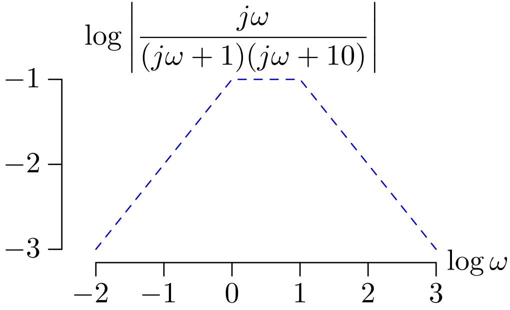




Bode Plot: Adding Instead of Multiplying

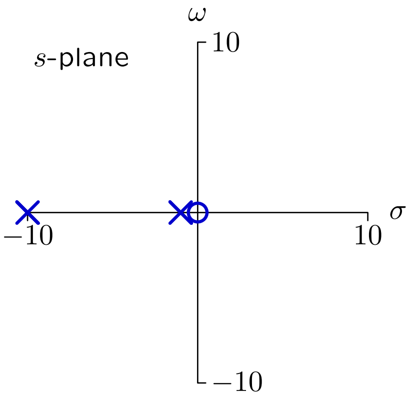
$$H(s) = \frac{s}{(s+1)(s+10)}$$

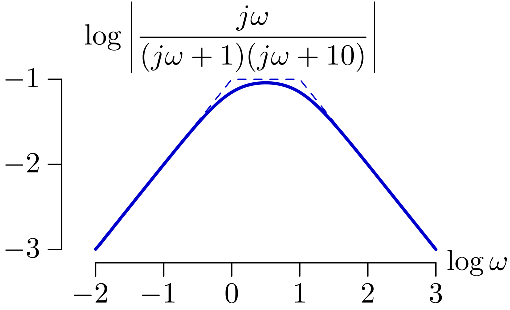




Bode Plot: Adding Instead of Multiplying

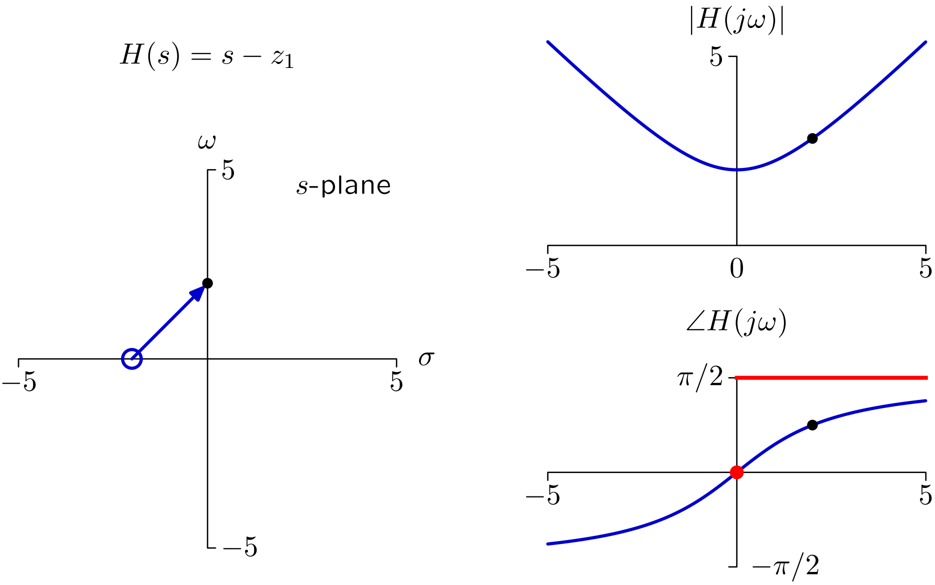
$$H(s) = \frac{s}{(s+1)(s+10)}$$





Asymptotic Behavior: Single Zero

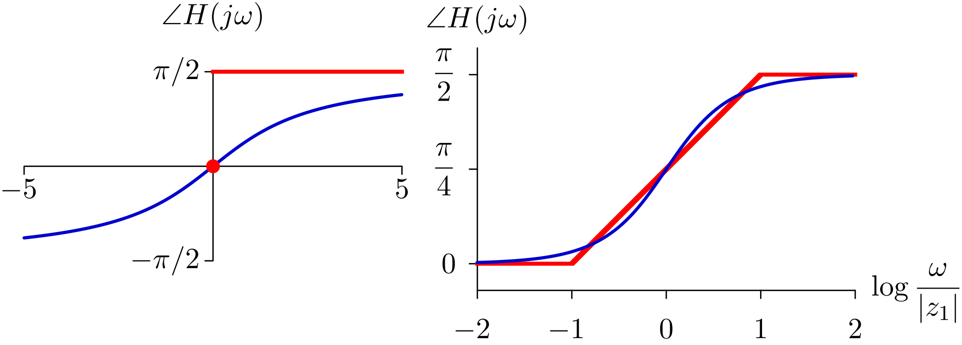
The angle response is simple at low and high frequencies.



Asymptotic Behavior: Single Zero

Three straight lines provide a good approximation versus log ω.

$$H(s) = s - z_1$$



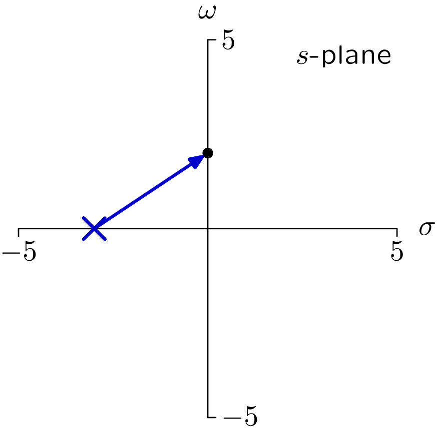
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

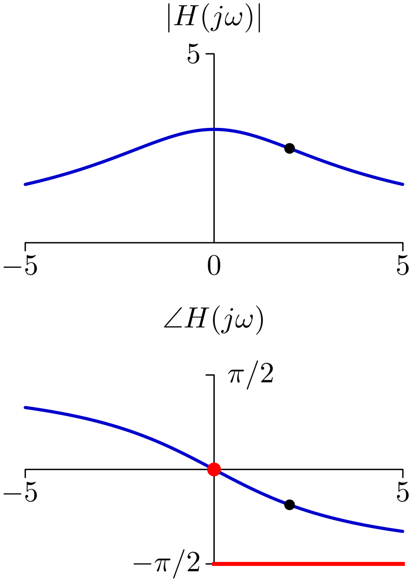
$$\lim_{\omega \to \infty} \angle H(j\omega) = \pi/2$$

Asymptotic Behavior: Single Pole

The angle response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$

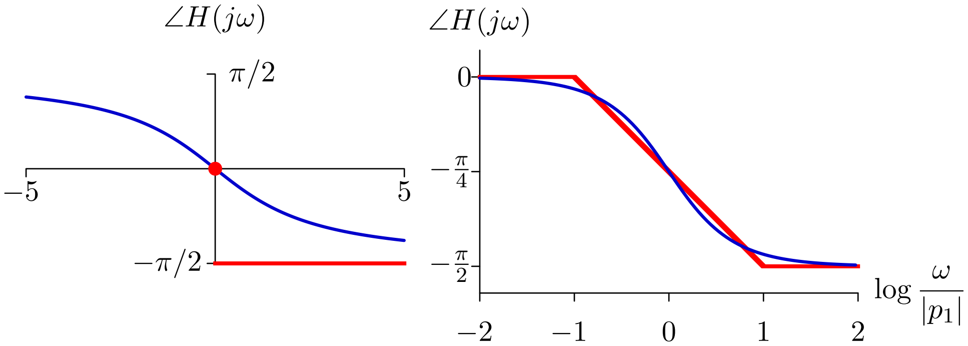




Asymptotic Behavior: Single Pole

Three straight lines provide a good approximation versus log ω.

$$H(s) = \frac{9}{s - p_1}$$



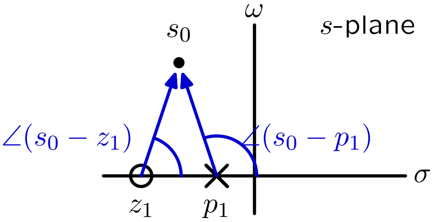
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = -\pi/2$$

Bode Plot

The angle of a product is the sum of the angles.

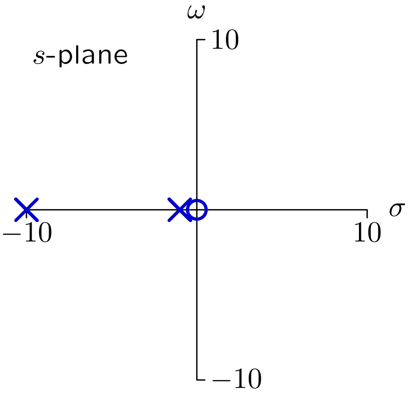
$$\angle H(s_0) = \angle\left(K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right) = \angle K + \sum_{q=1}^{Q} \angle(s_0 - z_q) - \sum_{p=1}^{P} \angle(s_0 - p_p)$$

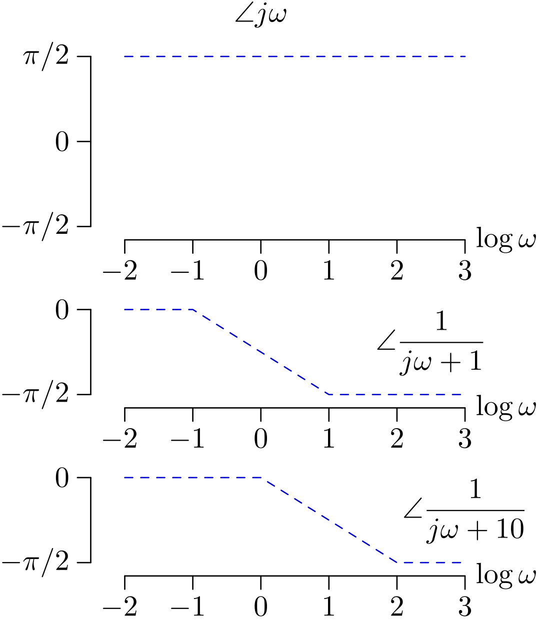


The angle of K can be 0 or π for systems described by linear differential equations with constant, real-valued coefficients.

Bode Plot

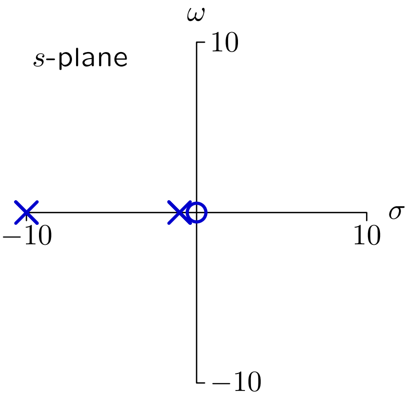
$$H(s) = \frac{s}{(s+1)(s+10)}$$

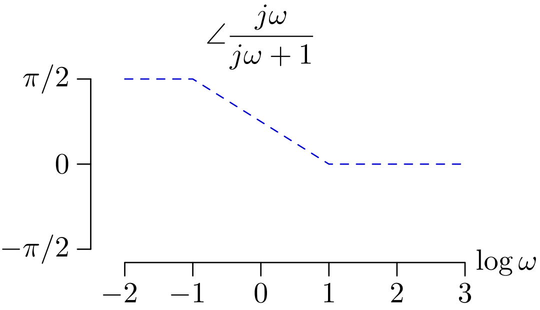


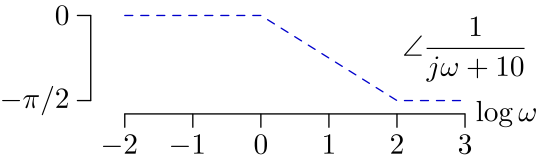


Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$

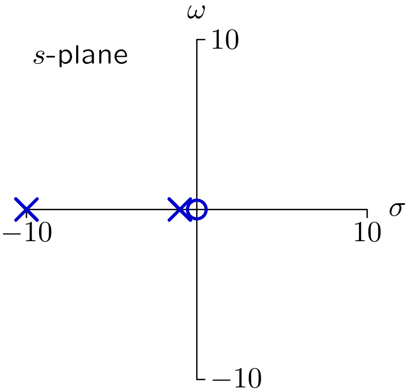


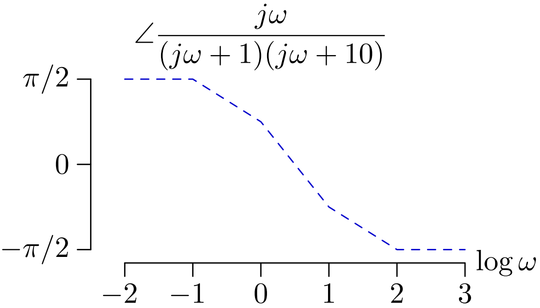




Bode Plot

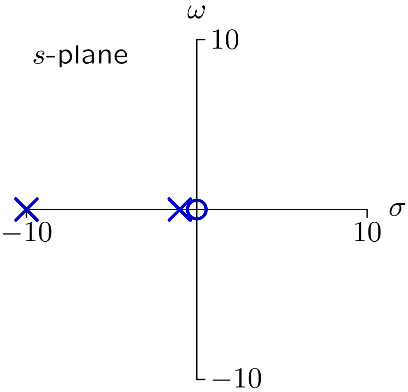
$$H(s) = \frac{s}{(s+1)(s+10)}$$

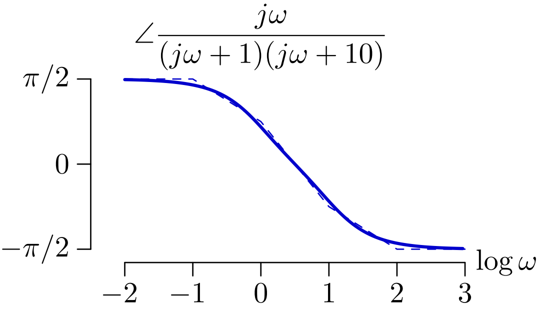




Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$





From Frequency Response to Bode Plot

The magnitude of $H(j\omega)$ is a product of magnitudes.

$$|H(j\omega)| = |K| \, \frac{\prod_{q=1}^{Q} |j\omega - z_q|}{\prod_{p=1}^{P} |j\omega - p_p|}$$

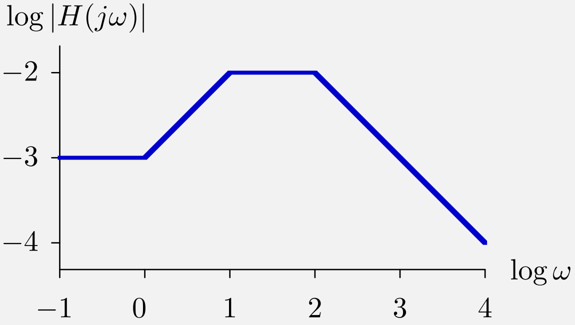
The log of the magnitude is a sum of logs.

$$\log|H(j\omega)| = \log|K| + \sum_{q=1}^{Q} \log|j\omega - z_q| - \sum_{p=1}^{P} \log|j\omega - p_p|$$

The angle of $H(j\omega)$ is a sum of angles.

$$\angle H(j\omega) = \angle K + \sum_{q=1}^{Q} \angle(j\omega - z_q) - \sum_{p=1}^{P} \angle(j\omega - p_p)$$

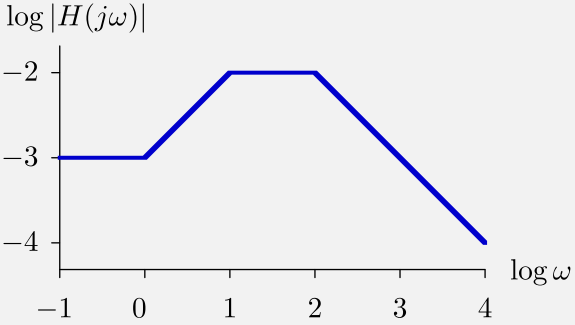
Check Yourself



Which corresponds to the Bode approximation above?

1. $\dfrac{1}{(s+1)(s+10)(s+100)}$
2. $\dfrac{s+1}{(s+10)(s+100)}$
3. $\dfrac{(s+10)(s+100)}{s+1}$
4. $\dfrac{s+100}{(s+1)(s+10)}$
5. none of the above

Check Yourself



Which corresponds to the Bode approximation above? 2

1. $\frac{1}{(s+1)(s+10)(s+100)}$

2. $\frac{s+1}{(s+10)(s+100)}$

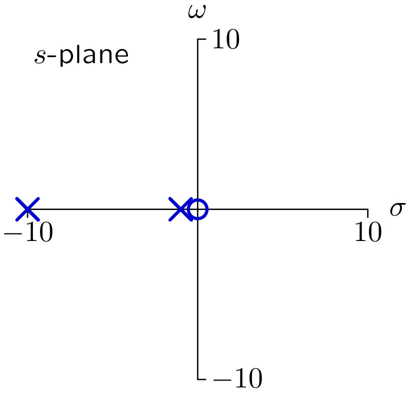
3. $\frac{(s+10)(s+100)}{s+1}$

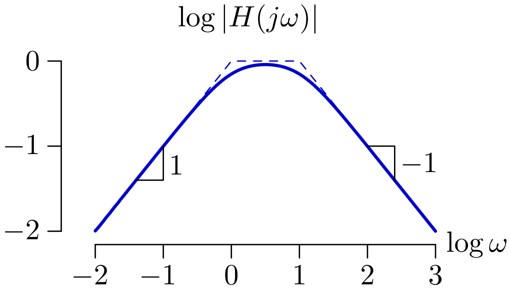
4. $\frac{s+100}{(s+1)(s+10)}$

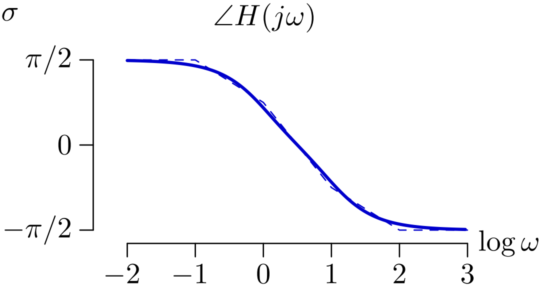
5. none of the above

Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$

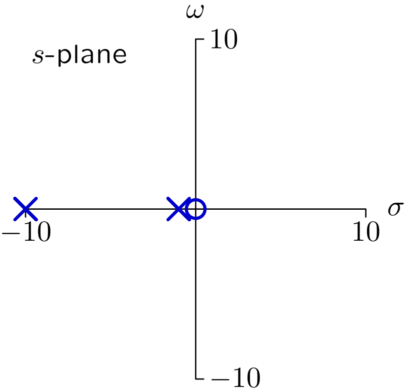


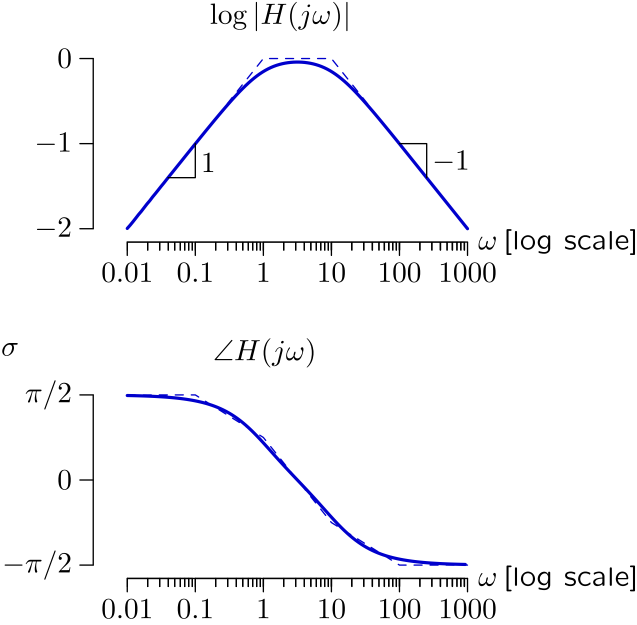




Bode Plot: dB

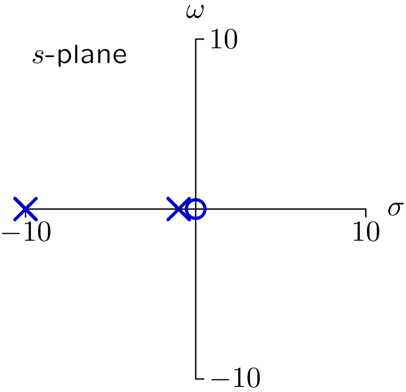
$$H(s) = \frac{10s}{(s+1)(s+10)}$$

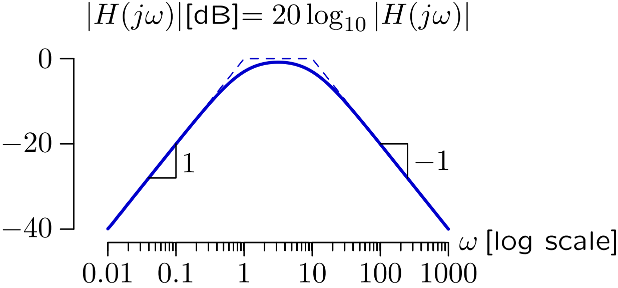


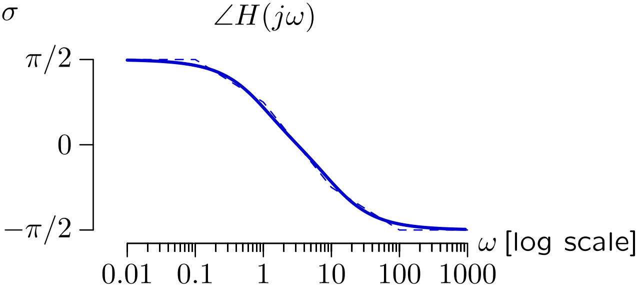


Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$

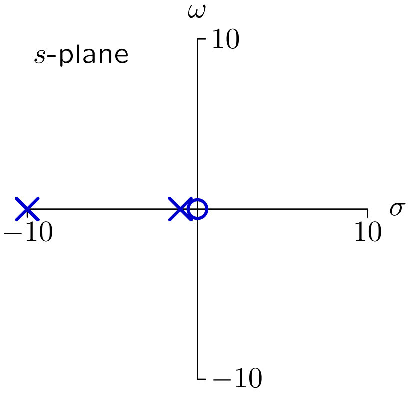


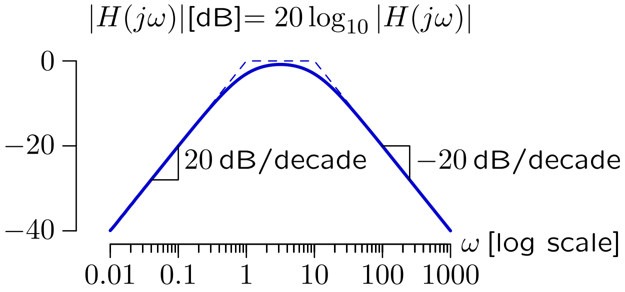


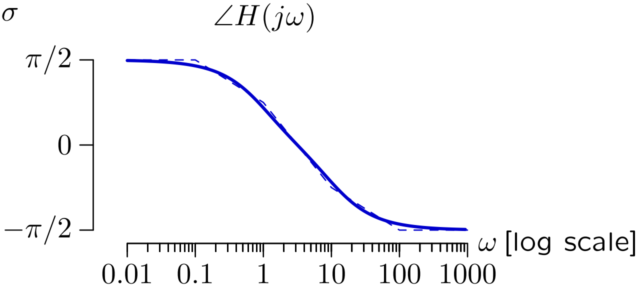


Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$

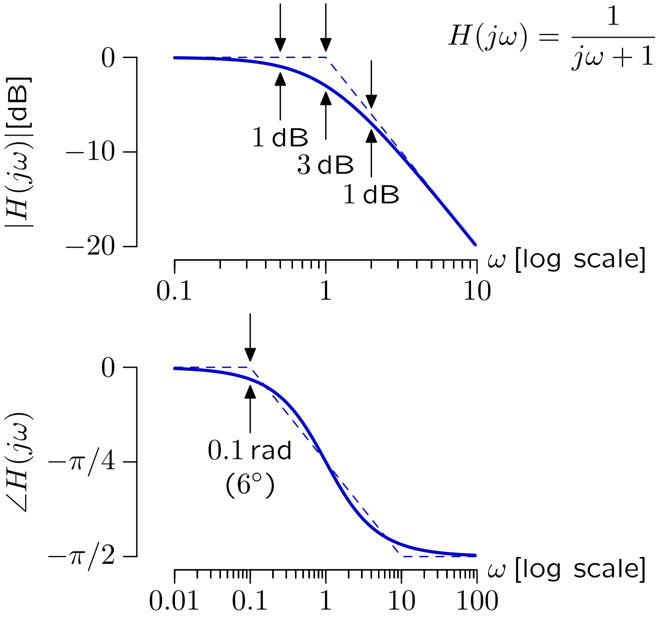






Bode Plot: Accuracy

The straight-line approximations are surprisingly accurate.



$$H(j\omega) = \frac{1}{j\omega + 1}$$

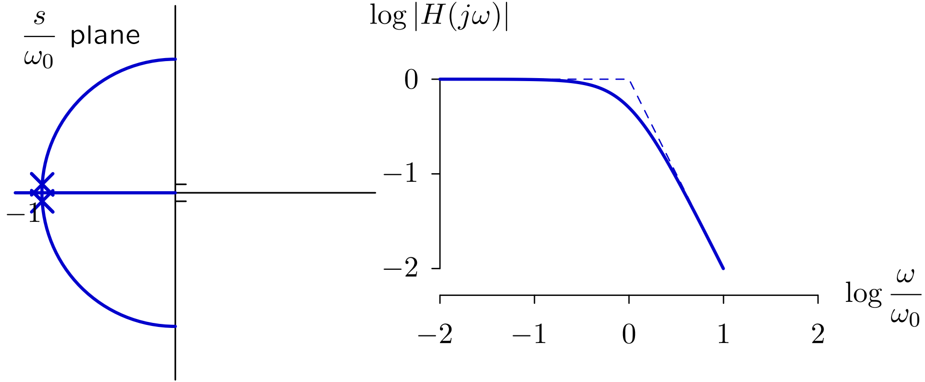
X	$20\log_{10} X$
1	0 dB
$\sqrt{2}$	≈ 3 dB
2	≈ 6 dB
10	20 dB
100	40 dB

Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 0.501$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

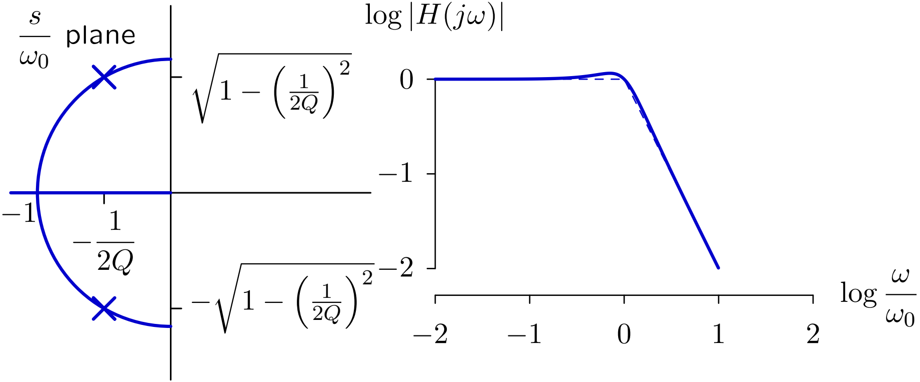


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$Q = 1$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

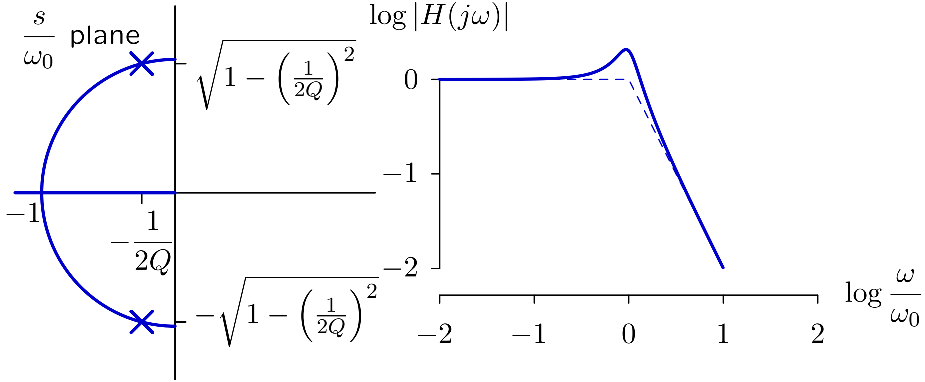


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 2$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

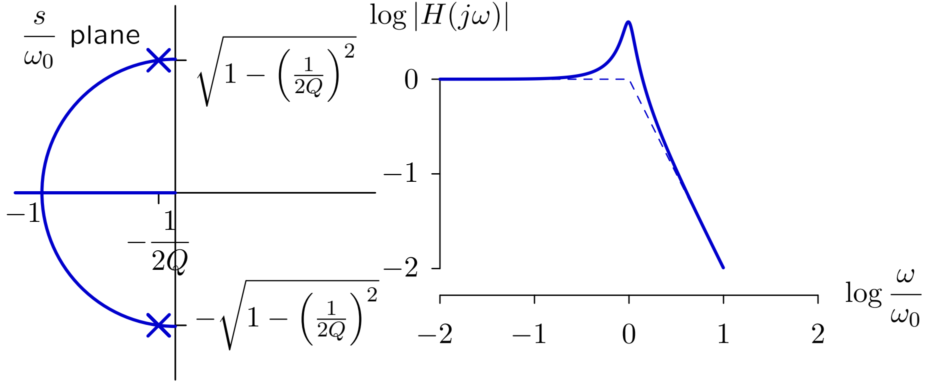


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 4$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

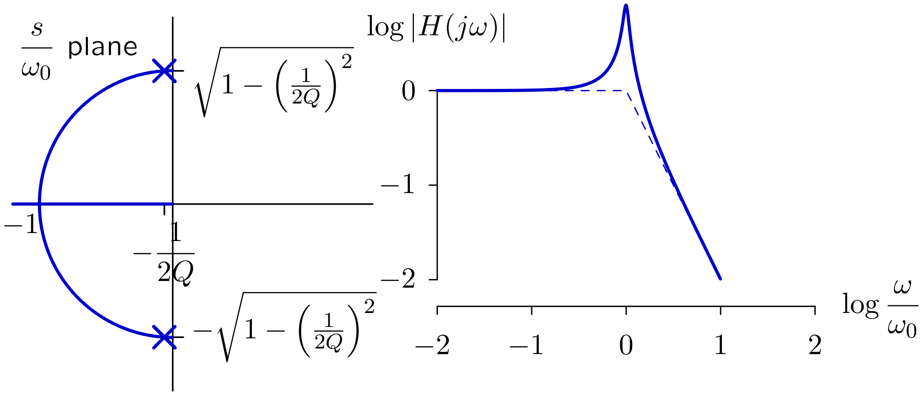


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 8$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

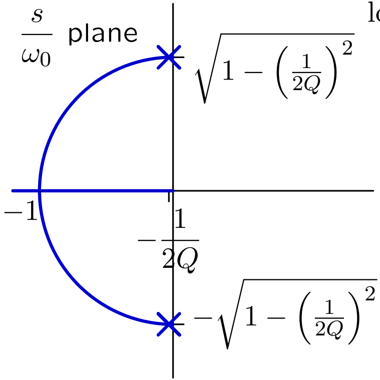


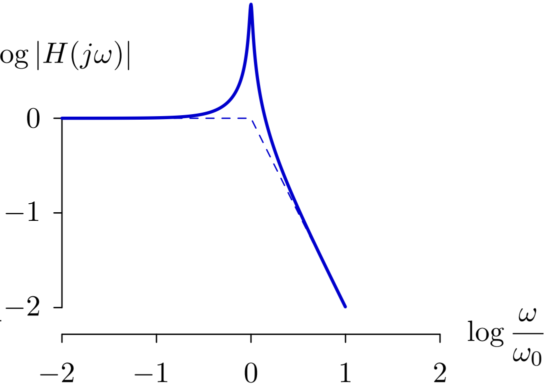
Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 16$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



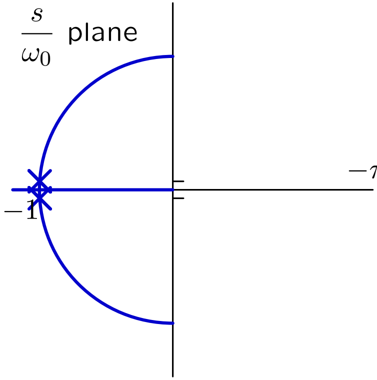


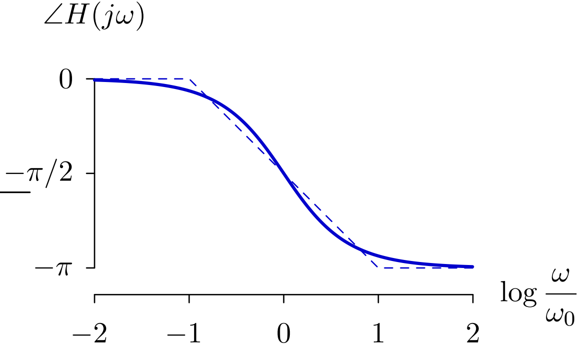
Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$Q = 0.501$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



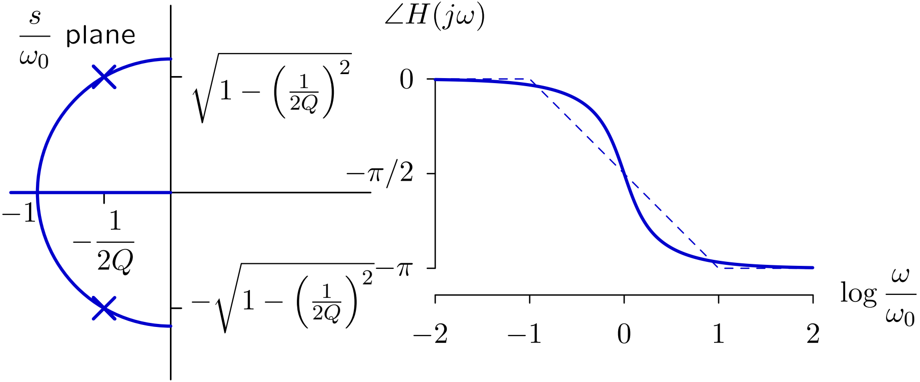


Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$Q = 1$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

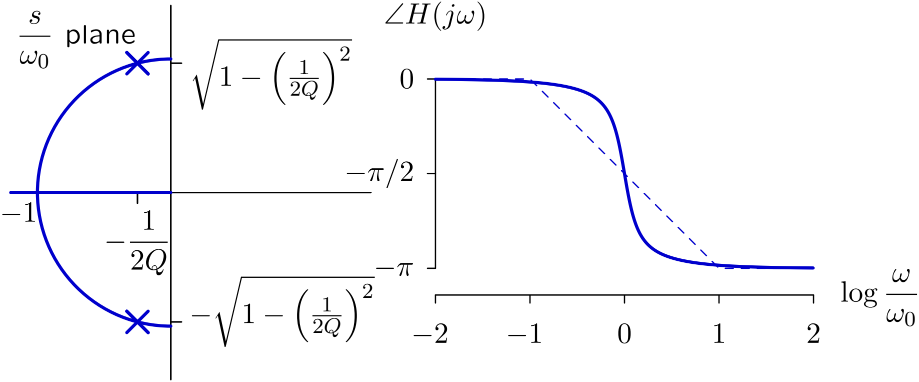


Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$Q = 2$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

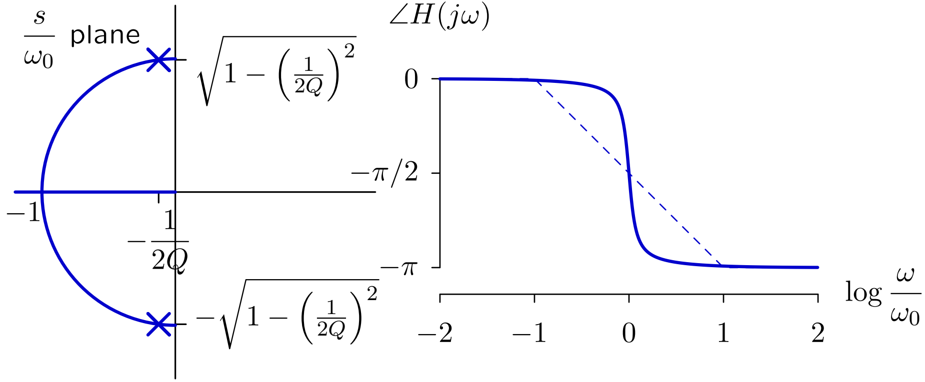


Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$Q = 4$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

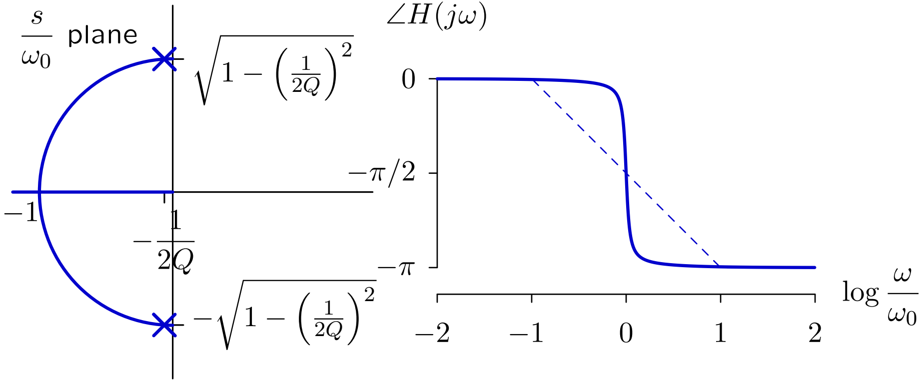


Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$Q = 8$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



Check Yourself

Determine Bode diagrams (magnitude and phase) for the following system function:

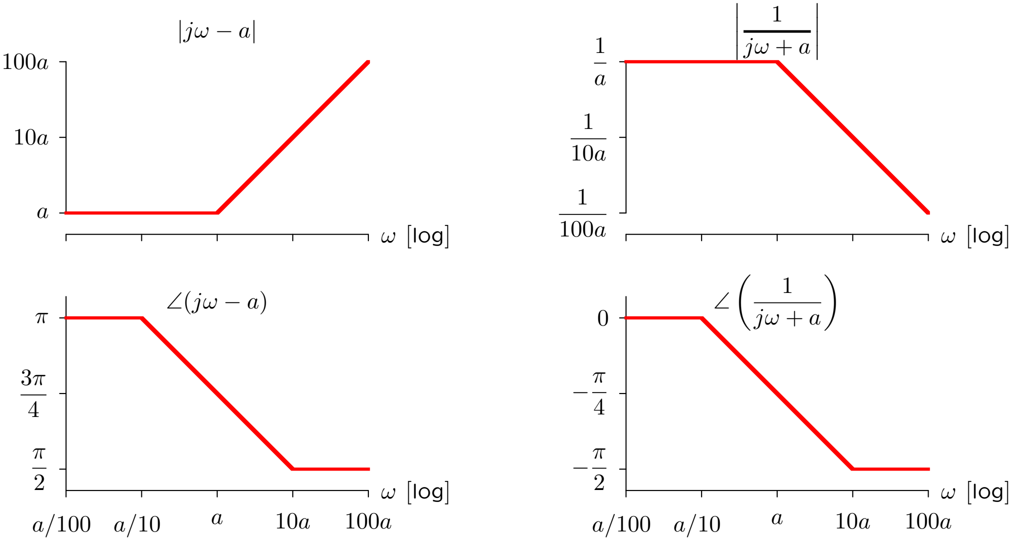
$$H(s) = \frac{s-a}{s+a}$$

where a is a positive, real number.

Check Yourself

Determine Bode diagrams (magnitude and phase) for the following system function:

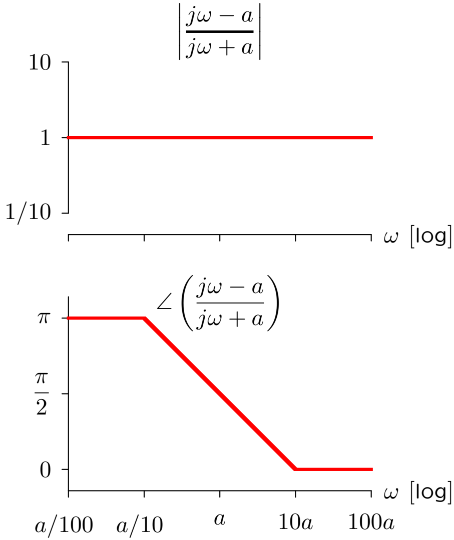
$$H(s) = \frac{s-a}{s+a}$$



Check Yourself

Determine Bode diagrams (magnitude and phase) for the following system function:

$$H(s) = \frac{s-a}{s+a}$$

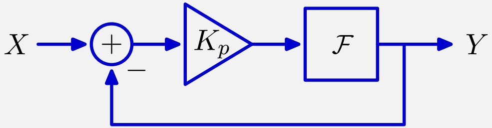


Parameter Estimation

Application of frequency response in **parameter estimation**.

Check Yourself

A proportional control scheme is used in the following system:



When $K_p = 1$, the closed-loop system function is given by

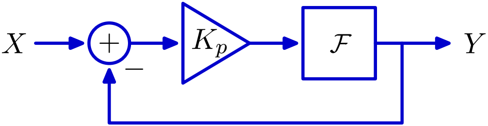
$$H(s) = \frac{Y}{X} = \frac{s+\alpha}{s^2+s+\alpha}$$

Determine the system function $\mathcal{F}$ for the plant.

1. $\frac{s+\alpha}{s^2}$
2. $\frac{s+\alpha}{s^2+s+\alpha}$
3. $\frac{s+\alpha}{s^2+2s+2\alpha}$
4. $\frac{s+\alpha}{s^2-s-\alpha}$
5. none of the above

Check Yourself

A proportional control scheme is used in the following system:



The general relation between $\mathcal{F}$ and $\mathcal{H}$ is given by Black's equation:

$$\mathcal{H} = \frac{K_p\mathcal{F}}{1 + K_p\mathcal{F}}$$

As indicated above, Black's equation is normally used to determine $\mathcal{H}$ from $\mathcal{F}$. However, Black's equation can be used to go in the opposite direction.

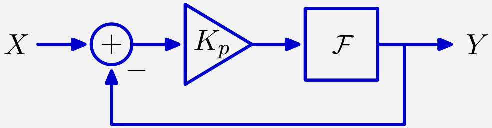
$$\mathcal{H} + K_p\mathcal{F}\mathcal{H} = K_p\mathcal{F}$$

$$\mathcal{F} = \frac{1}{K_p}\left(\frac{\mathcal{H}}{1-\mathcal{H}}\right) = \frac{\mathcal{H}}{1-\mathcal{H}} \quad \text{since } K_p = 1$$

If $\mathcal{H} = \dfrac{s+\alpha}{s^2+s+\alpha}$ then $\mathcal{F} = \dfrac{\left(\frac{s+\alpha}{s^2+s+\alpha}\right)}{\left(1 - \frac{s+\alpha}{s^2+s+\alpha}\right)} = \left(\dfrac{s+\alpha}{s^2}\right)$

Check Yourself

A proportional control scheme is used in the following system:



When $K_p = 1$, the closed-loop system function is given by

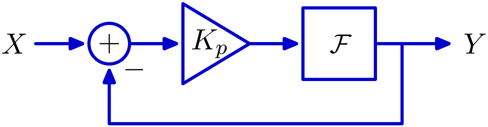
$$H(s) = \frac{Y}{X} = \frac{s+\alpha}{s^2+s+\alpha}$$

Determine the system function $\mathcal{F}$ for the plant. 1

1. $\dfrac{s+\alpha}{s^2}$
2. $\dfrac{s+\alpha}{s^2+s+\alpha}$
3. $\dfrac{s+\alpha}{s^2+2s+2\alpha}$
4. $\dfrac{s+\alpha}{s^2-s-\alpha}$
5. none of the above

Check Yourself

Notice that the system function for the plant is simpler than that of the closed loop system.



$$F = \frac{s+\alpha}{s^2}$$

$$H = \frac{Y}{X} = \frac{s+\alpha}{s^2+s+\alpha}$$

This motivates a method to estimate the parameters of a system from the closed-loop frequency response.

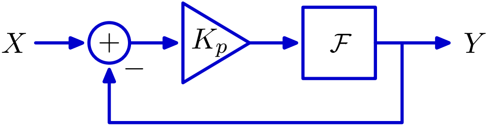
Parameter Estimation

Assume that we know the form of $F(s)$ from a model of the system:

$$F(s) = \frac{s+\alpha}{s^2}$$

but we don't know α.

Estimation procedure: measure the closed-loop frequency response of the system $\widetilde{H}(j\omega)$ when $K_p = 1$.



Compute the measured $\widetilde{F}(j\omega)$ from the measured $\widetilde{H}(j\omega)$:

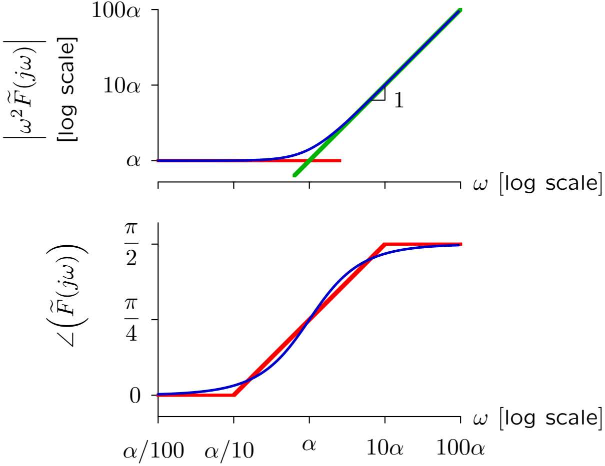
$$\widetilde{F}(j\omega) = \frac{1}{K_p}\left(\frac{\widetilde{H}(j\omega)}{1-\widetilde{H}(j\omega)}\right) = \frac{\widetilde{H}(j\omega)}{1-\widetilde{H}(j\omega)}$$

Compare the resulting frequency response for $\widetilde{F}(j\omega)$ with that of the model:

$$\widetilde{F}(j\omega) = \frac{j\omega+\alpha}{(-j\omega)^2} = \frac{j\omega+\alpha}{-\omega^2}$$

Parameter Estimation

From this analysis, we expect that $-\omega^2\widetilde{F}(j\omega)$ will have the following form: $j\omega+\alpha$ (a single zero).

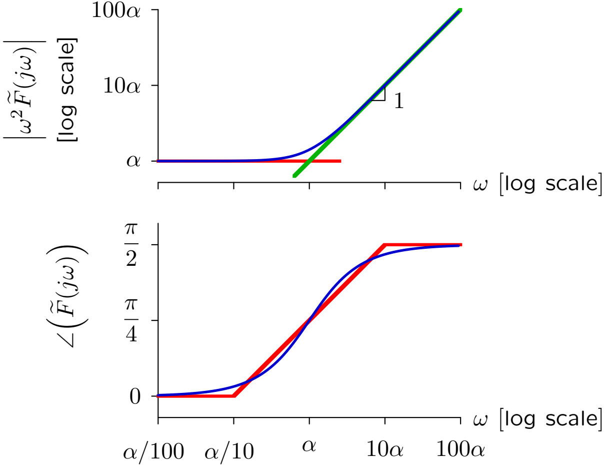


We can estimate the value of the parameter α

- from the low-frequency gain, or
- from the frequency where the two magnitude asymptotes intersect, or
- from the frequency where the phase is $\pi/4$.

Parameter Estimation

From this analysis, we expect that $-\omega^2 \widetilde{F}(j\omega)$ will have the following form: $j\omega + \alpha$ (a single zero).



We will use a variant of this method to estimate the parameters of the two-propeller arm in this week's postlab.

Summary

The frequency response of a system is easily determined using Bode plots.

Each pole and each zero contributes one section to the Bode plot.

The magnitude of the response of the system is given by the sum of the magnitudes for the sections contributed by each pole and zero.

The angle of the response of the system is given by the sum of the angles for the sections contributed by each pole and zero.