April 11, 2023

6.3100 Lecture 17 Notes – Spring 2023

Connection between Continuous Time and Discrete Time State-Space Control

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Outline:

1. Discrete time state space control
2. Eigenvalue, stability, and spectral theorem
3. Example: scalar system
4. Example: pendulum

1. **Discrete time state space control**

Since last week, we introduced state space control in the context of continuous time systems. This is true for most physical systems because they are characterized by differential equations. However, most physical systems are controlled by microprocessors that run at a discrete clock cycle. Practically, it is important to learn about the connections between discrete time and continuous time systems.

The general form of continuous time state-space system is:

$$E\,\dot{x}(t) = Ax(t) + Bu(t)$$

$$y = Cx(t)$$

Suppose we want to convert these equations to discrete time. We first convert the input, state, and output variables:

$$\begin{aligned} u[n] &= u(n\Delta T) \\ x[n] &= x(n\Delta T) \\ y[n] &= y(n\Delta T) \end{aligned}$$

The two systems are identical if the input signal is piecewise constant: $u(t) = u[floor(\frac{t}{\Delta T})]$.

We want to rewrite the discrete time system in the following form:

$$\begin{aligned} x[n] &= A_d x[n-1] + B_d u[n-1] \\ y[n] &= C_d x[n] \end{aligned}$$

Note that here we “hide” the matrix E_d by simply letting it be an identity matrix. This is more of a computational convenience. In practice, as long as E_d is invertible, we can multiply the equation by E_d^{-1} to remove this term.

The key question is how to solve for the equivalent $A_d, B_d, and\ C_d$?

For the CT system with initial condition x(0) and constant input u(0), the analytical solution is:

$$x(t) = e^{E^{-1}At}x(0) + \left(e^{E^{-1}At} - I\right)A^{-1}Bu(0)$$

If we take two specific time $n\Delta T$ and $(n-1)\Delta T$, then the relationship is given by:

$$x(n\Delta T) = e^{E^{-1}A\Delta T}x\big((n-1)\Delta T\big) + \big(e^{E^{-1}A\Delta T} - I\big)A^{-1}Bu((n-1)\Delta T)$$

Now we can substitute the DT variables:

$$x[n] = e^{E^{-1}A\Delta T}x[n-1] + \big(e^{E^{-1}A\Delta T} - I\big)A^{-1}Bu[n-1]$$

The output to state variable relationship is unchanged:

$$y(t) = Cx(t) \quad \rightarrow \quad y[n] = C_d x[n]$$

Matching terms we will have:

$$A_d = e^{E^{-1}A\Delta T}$$
$$B_d = \big(e^{E^{-1}A\Delta T} - I\big)A^{-1}B$$
$$C_d = C$$

This is the analytical conversion equation. Practically, calculating matrix exponential can be expensive. In many cases, we can approximate the using Taylor series expansion:

$$A_d = e^{E^{-1}A\Delta T} \approx I + E^{-1}A\Delta T$$
$$B_d = (e^{E^{-1}A\Delta T} - I)A^{-1}B \approx E^{-1}B\Delta T$$
$$C_d = C$$

It is much easier to calculate matrix products than matrix exponentials.

Finally, for completeness, we give the general solution of DT system without proof:

$$y[k] = C_d A_d^k x[0] + C_d \sum_{i=0}^{k-1} A_d^{k-i-1} B_d u[k]$$

2. Eigenvalue, stability, and spectral theorem

Next, we analyze the open-loop stability property of DT system. To analyze open-loop stability, we need to relate input u with output y. We perform z-transform:

$$X = A_d X z^{-1} + B_d U z^{-1} \rightarrow X = (zI - A_d)^{-1} B_d U$$
$$Y = C_d X = C_d (zI - A_d)^{-1} B_d$$

The open-loop transfer function is given by:

$$H_{open}(z) = C_d (zI - A_d)^{-1} B_d$$

From the result presented from the previous lecture, the poles are given by the eigenvalues of the matrix A_d, which determines open loop stability. How do we calculate the eigenvalue of A_d? Keep in mind that we are converting from a CT system, so most likely we know the eigenvalues of A. The core question becomes how to figure out the eigenvalues of A_d from the eigenvalues of A.

We invoke the spectral theorem in linear algebra. The spectral theorem states that given $f(z) = \sum_{l=0}^{L} a_l z^l$, if λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$. This condition gives us a simple result:

If λ is an eigenvalue of A, then $e^{E^{-1}\lambda\Delta T}$ is an eigenvalue of $e^{E^{-1}A\Delta T}$, which is A_d.

For CT system, the system is stable if max(real(λ_A)) < 0. That is, all eigenvalues of A must be in the left half plane. For DT system, the system is stable if $|\lambda_{A_d}| < 1$. That is, all eigenvalues of A_d must be within the unit circle.

Next, we will work through 2 examples to analyze system stability.

3. Example: scalar system

Consider the scalar state space system below:

$$3\frac{dx(t)}{dt} = -25x(t) + 15u(t)$$

$$y(t) = x(t)$$

1) Please convert it into discrete time system with $\Delta T = 1/20$.

Solution: We have E = 3, A = -25, B = 15, C = 1, D = 0. We can use the analytical equations or first-order Taylor series expansion equations. Here we show some MATLAB commands.

We can formulate a continuous time system: C_sys = dss(A,B,C,D,E);

Then we set dt = 1/20;

Next, we convert the system into a DT system: D_sys = c2d(C_sys,dt);

We can print out the converted matrices by typing D_sys.A, etc.

2) What are the eigenvalues of exact and approximate DT systems.

Exact system: D_pole = eig(D_sys.A), or eig(expm(E^(-1)* A*dt));

Approximate system: D_pole_approx = eig(1 + dt * E^-1*A);

MATLAB will return D_pole = 0.6592 and D_pole_approx = 0.5833

3) For what ΔT value will the exact DT eigenvalue change to 0.99?

We need to solve for $0.99 = e^{E^{-1}A\Delta T}$. This is given by $\Delta T = \frac{log 0.99}{E^{-1}A}$.

4. Example: pendulum

Let's consider a matrix problem that has two states: $\theta(t)\ and\ w(t)$. This system of equation models a pendulum. It is very similar to the inverted pendulum example discussed in the previous lecture. For a pendulum system, the state vector is defined as:

$$x(t) = \begin{bmatrix} \theta(t) \\ w(t) \end{bmatrix}$$

The CT equation is given by:

$$\frac{d}{dt}\begin{bmatrix} \theta(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} \theta(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$

$$\theta(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \theta(t) \\ w(t) \end{bmatrix}$$

1. Please find the DT system representation, A_d, B_d, and C_d. Here dt = 1/20.

$$A_d = e^{E^{-1}A\Delta T} = \begin{bmatrix} 0.9888 & 0.0498 \\ -0.4483 & 0.9888 \end{bmatrix}$$

$$B_d = \left(e^{E^{-1}A\Delta T} - I\right)A^{-1}B = \begin{bmatrix} 0.0025 \\ 0.0996 \end{bmatrix}$$

$$C_d = C = [1\ 0]$$

2. Find the eigenvalues of the CT and DT systems. Comment of system stability.

$$eig(A) = [-3j, 3j]$$

$$eig(A_d) = [0.9888 + 0.1494j, 0.9888 - 0.1494j]$$

Both systems are marginally stable.

3. What is the smallest ΔT such that the DT system's A_d matrix is the negative of an identity matrix?

This is an exercise of using the spectral theorem. We want:

$$eig\left(e^{E^{-1}A\Delta T}\right) = -1 = e^{\pm j\pi}$$

We know that

$$eig(E^{-1}A) = \pm 3j$$

Accordingly to the spectral theorem, we have

$$e^{(\pm 3j)\Delta T} = e^{\pm j\pi}$$

Solving for this relationship we will arrive at:

$$\Delta T = \frac{\pi}{3}$$

4. What is the stability property of an approximated DT system (using Taylor series expansion)?

We have:

$$A_{d,approx} = I + E^{-1}A\Delta T$$

Using the spectral theorem,

$$eig\left(A_{d,approx}\right) = 1 + eig(A)\Delta T = 1 \pm 3j\Delta T$$

Note that the magnitude of $|1 \pm 3j\Delta T| > 1$, which means the approximate system is unstable! The properties of approximated DT system may not be stable!