6.3100: Dynamic System Modeling and Control Design

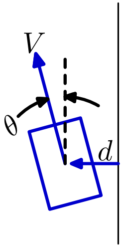
Discrete-Time System Functions

February 26, 2024

Modeling Systems with Difference Equations

Over the past several weeks, we have seen many examples of how difference equations can be used to describe and improve the behaviors of systems.

Example: **robotic steering**



$$d[n] = d[n-1] + V\Delta T\theta[n-1]$$

$$\theta[n] = \theta[n-1] + \Delta T\omega[n-1]$$

$$\omega[n] = \gamma u[n]$$

$$u[n-1] = K_p(d_d[n-1] - d[n-1]$$

Simple (but somewhat tedious) math yields a difference equation that relates the input $d_d[\cdot]$ and output $d[\cdot]$:

$$d[n] = 2d[n-1] - d[n-2] + (\Delta T)^2 V K_p \gamma\Big(d_d[n-2] - d[n-2]\Big)$$

which can then be analyzed to gain insight into behaviors of the system.

Insights from Difference Equations

The difference equation can be used to find a system's response to any **arbitrary** input $d_d[n]$.

Start from an **initial condition**, e.g.,

$$d[n] = 0 \text{ for } n < 0$$

Then **step** through n, using the difference equation

$$d[n] = 2d[n-1] - d[n-2] + (\Delta T)^2 V K_p \gamma \Big(d_d[n-2] - d[n-2] \Big)$$

to calculate successive values of $d[n]$ from the input $d_d[\,\cdot\,]$ and previous values of the output ($d[n-1]$ and $d[n-2]$).

$\rightarrow$ useful for characterizing responses to a specified input signal $d_d[n]$, e.g., the step response.

Insights from Difference Equations

The difference equation can also be used to find the **natural frequencies** of a system.

Find the value or values of λ for which $d[n] = \lambda^n$ is a solution to the difference equation when $d_d[n] = 0$ (i.e., the homogeneous case).

$$d[n] = 2d[n-1] - d[n-2] + (\Delta T)^2 V K_p \gamma \Big(d_d[n-2] - d[n-2] \Big)$$

$$\lambda^n - 2\lambda^{n-1} + (1+(\Delta T)^2 V K_p \gamma)\lambda^{n-2} = 0$$

$$\lambda^2 - 2\lambda + 1+(\Delta T)^2 V K_p \gamma = 0$$

$$\lambda = 1 \pm j\Delta T \sqrt{V K_p \gamma}$$

$\rightarrow$ useful for characterizing performance metrics (stability, convergence, etc.) of a system without having to specify the signals that excite them.

Modeling Systems with Difference Equations

Difference eqn's provide two different but closely related views of a system.

Time domain: step-by-step calculation of **samples**:

$$d[n] = 2d[n-1] - d[n-2] + (\Delta T)^2 V K_p \gamma \Big(d_d[n-2] - d[n-2] \Big)$$

Frequency domain: constraints on the structure of the output **signal**:

$$\lambda^n = 2\lambda^{n-1} - \lambda^{n-2} - (\Delta T)^2 V K_p \gamma \lambda^{n-2}$$

While there are considerable differences between these views, their underlying structures are surprisingly similar. And the similarities are even more striking when expressed as block diagrams.

Modeling Systems with Difference Equations

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Time domain: step-by-step calculation of **samples**:

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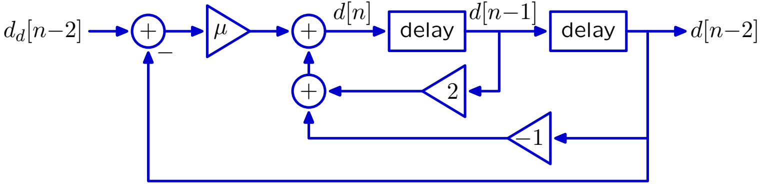

Frequency domain: constraints on the structure of the output **signal**:

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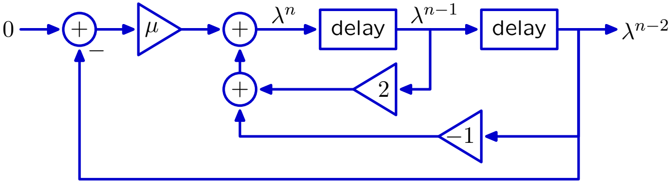


Same structures – only the labels are different.

Time and Frequency Domain Methods

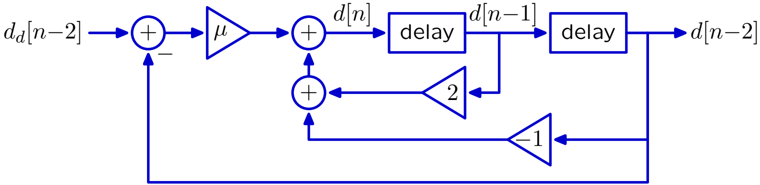
We can exploit relations between time and frequency domain formulations to simplify our work and deepen our understanding of control systems.

We begin by casting the two formulations into a common framework.

Polynomial (Functional) Representations

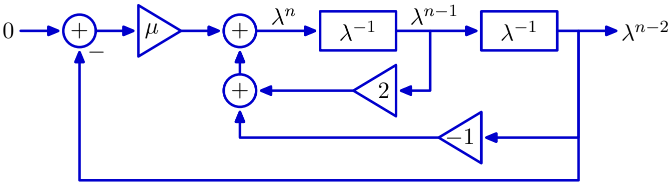
The function of the **delay** box is clear for the time-domain representation.

$$d[n] = 2d[n-1] - d[n-2] + (\Delta T)^2 V K_p \gamma \Big(d_d[n-2] - d[n-2] \Big)$$



The function of the **delay** box is a bit different in the frequency domain.

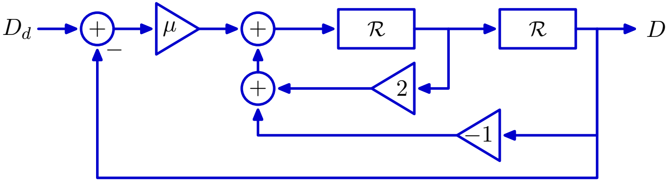
$$\lambda^n = 2\lambda^{n-1} - \lambda^{n-2} - (\Delta T)^2 V K_p \gamma \lambda^{n-2}$$



Delaying λ^n by one sample in time is equivalent to multiplying the entire signal by a constant (λ^{-1}). Geometric signals are **eigenfunctions**.

Polynomial (Functional) Representations

Let $\mathcal{R}$ represent a generic **operator** that can represent delay in the time domain or multiplication by inverse frequency in the frequency domain.



We can think of $\mathcal{R}$ as an operator. If X represents a signal $x[n]$, then $\mathcal{R}X$ represents a **right-shifted** version of X.

Operator Notation: Check Yourself

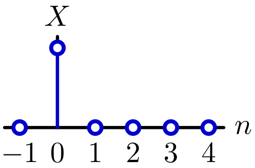
$$X \longrightarrow \boxed{\mathcal{R}} \longrightarrow Y = \mathcal{R}X$$

Let $Y = \mathcal{R}X$. Which of the following is/are true:

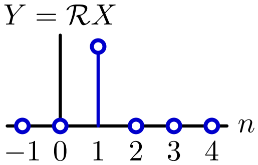
1. $y[n] = x[n]$ for all n
2. $y[n] = x[n-1]$ for all n
3. $y[n] = x[n+1]$ for all n
4. $y[n-1] = x[n]$ for all n
5. none of the above

Check Yourself

Consider a simple signal:



Then



Clearly $y[1] = x[0]$. Equivalently, if $n = 1$, then $y[n] = x[n-1]$.
The same sort of argument works for all other n.

Operator Notation: Check Yourself

$$X \longrightarrow \boxed{\mathcal{R}} \longrightarrow Y = \mathcal{R}X$$

Let $Y = \mathcal{R}X$. Which of the following is/are true: 2

1. $y[n] = x[n]$ for all n
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3. $y[n] = x[n+1]$ for all n
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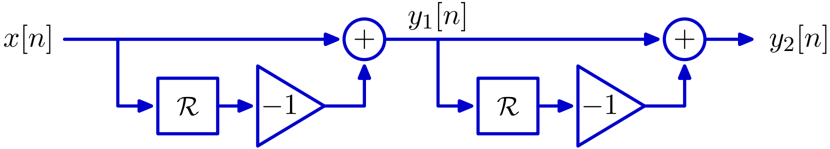
Polynomial (Functional) Representations

Instead of **difference equations** to specify relations among **samples**, we use **polynomials** in $\mathcal{R}$ to specify relations among entire **signals**.



Relations between **samples**.

$$\begin{aligned} y_2[n] &= y_1[n]-y_1[n-1] \\ &= (x[n]-x[n-1]) - (x[n-1]-x[n-2]) \\ &= x[n] - 2x[n-1] + x[n-2] \end{aligned}$$

Relations between **signals**.

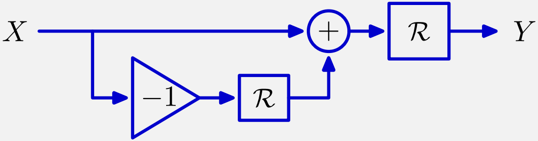
$$\begin{aligned} Y_2 &= (1-\mathcal{R})\{Y_1\} = (1-\mathcal{R})\{(1-\mathcal{R})\{X\}\} = (1-\mathcal{R})(1-\mathcal{R})\,X \\ &= (1-\mathcal{R})^2 X \\ &= (1-2\mathcal{R}+\mathcal{R}^2)\,X \end{aligned}$$

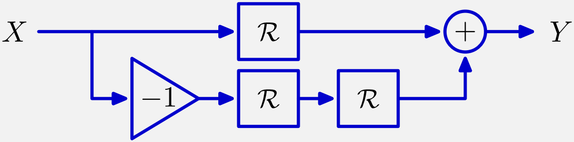
Notice that the $\mathcal{R}$ representation obeys familiar properties of **polynomials**.

Check Yourself

Operator expressions obey many of the algebraic rules of polynomials. The following systems are described by the same difference equation:

$$y[n] = x[n-1] - x[n-2]$$

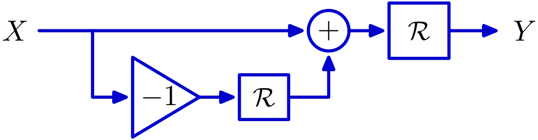




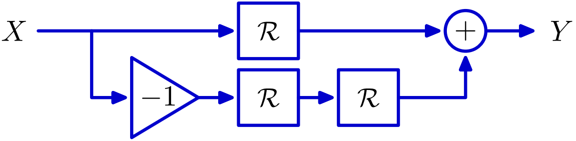
Their operator expressions are related by what math property?

1. commutativity
2. associativity
3. distributivity
4. transitivity
5. none of the above

Check Yourself



$$Y = \mathcal{R}(1 - \mathcal{R})X$$



$$Y = (\mathcal{R} - \mathcal{R}^2)X$$

Multiplication by $\mathcal{R}$ distributes over addition.

Check Yourself

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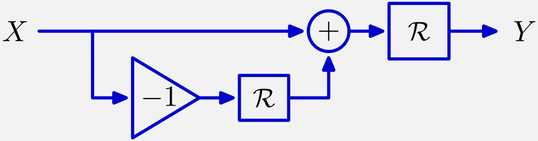
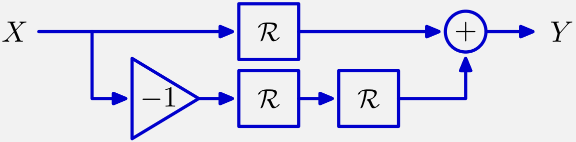
The following systems are described by the same difference equation:

$$y[n] = x[n-1] - x[n-2]$$





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1. commutativity
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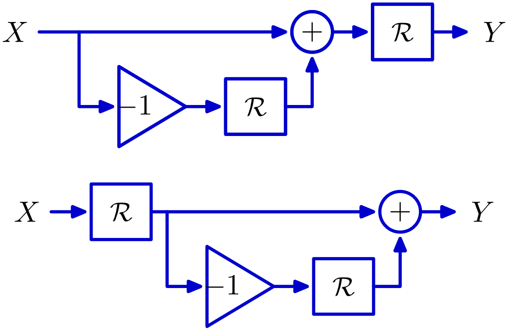
Operator Algebra

Similarly, operator expressions obey the commutativity principle:

$$\mathcal{R}(1-\mathcal{R})\,X = (1-\mathcal{R})\mathcal{R}X$$



These systems are equivalent in the sense that they are described by the same difference equation:

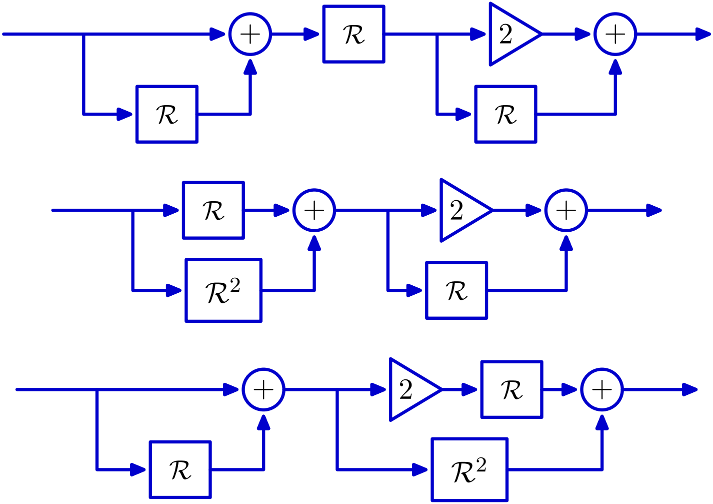
$$y[n] = x[n-1] - x[n-2]$$

Operator Algebra

The associative property similarly holds for operator expressions.

$$(2+\mathcal{R})\mathcal{R}(1+\mathcal{R}) = (2+\mathcal{R})\Big(\mathcal{R}(1+\mathcal{R})\Big) = \Big((2+\mathcal{R})\mathcal{R}\Big)(1+\mathcal{R})$$

Corresponding block diagrams:



Using Operators to Analyze Systems

The polynomial representation retains the structure of the underlying difference equations, and allows us to manipulate and simplify difference equations using **polynomial mathematics**.

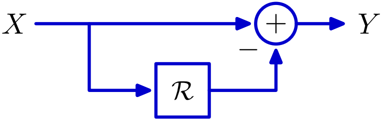
$\rightarrow$ polynomials are generally easier to work with than difference equations

$\rightarrow$ polynomials provide insights not apparent from difference equations

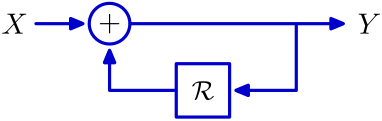
Next: using operators to analyze systems.

Feedforward and Feedback Pathways

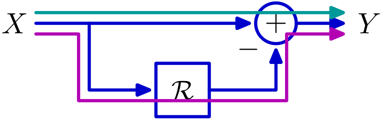
A **cyclic pathway** is one that closes a loop on itself.

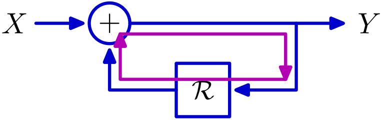


acyclic



cyclic



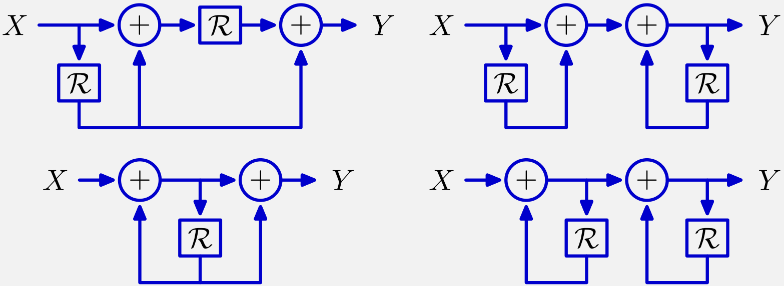


Feedforward systems contain no cyclic pathways. Their responses consist of a sum of components: each characterized by an aggregate gain and delay.

Feedback systems contain one or more cyclic pathways. Their responses can persist **long** after the input ends, as signals propagate through internal loops.

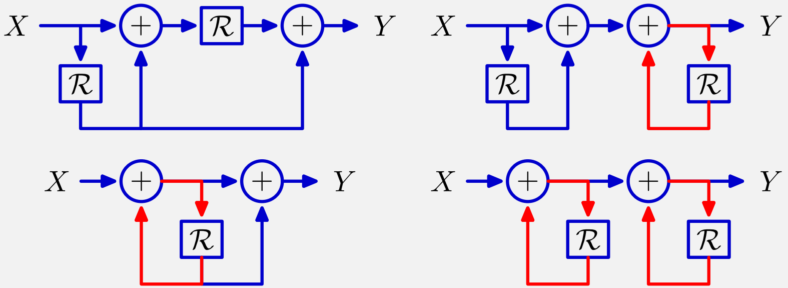
Check Yourself

How many of the following systems have cyclic signal paths?



Check Yourself

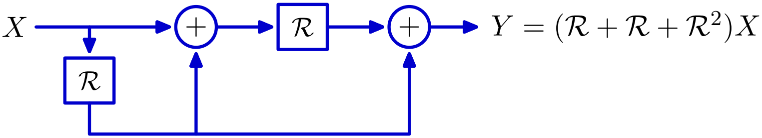
How many of the following systems have cyclic signal paths? 3



Using Operators to Analyze Feedforward Systems

Feedforward systems that are constructed of adders, gains, and delays can be represented by a polynomial in $\mathcal{R}$.

Example:



There are 3 pathways through this system: two have a single delay, and one has two delays.

Feedforward systems are characterized by a **functional** $\mathcal{F}(\mathcal{R})$ that operates on the input to produce the output:

$$Y = \mathcal{F}(\mathcal{R})X$$

where

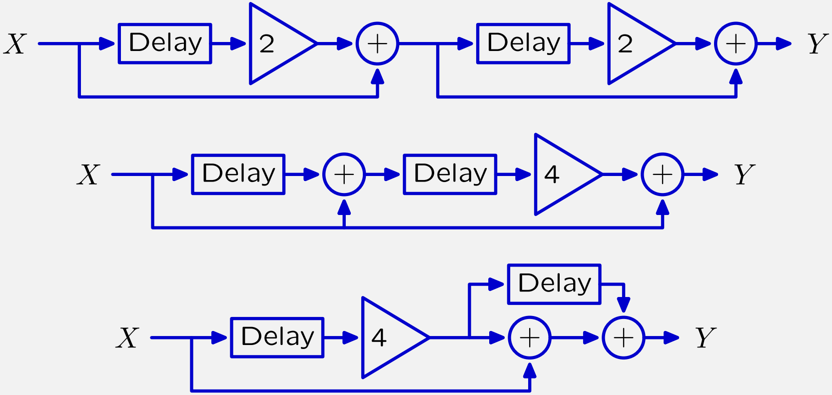
$$\mathcal{F}(\mathcal{R}) = \mathcal{R} + \mathcal{R} + \mathcal{R}^2$$

There is an **explicit** dependence of Y on X.

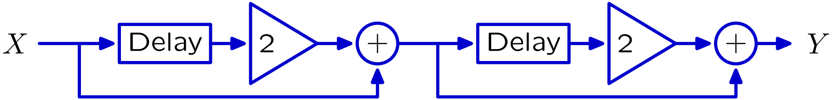
Check Yourself

How many of the following systems are equivalent to

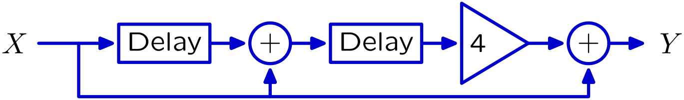
$$Y = (4\mathcal{R}^2 + 4\mathcal{R} + 1)\,X \quad ?$$



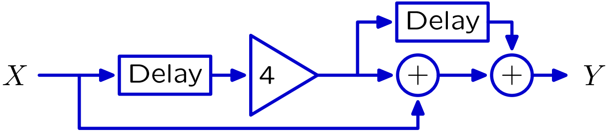
Check Yourself



$$Y = (2\mathcal{R}+1)(2\mathcal{R}+1)\,X$$



$$Y = (4\mathcal{R}^2+4\mathcal{R}+1)\,X$$



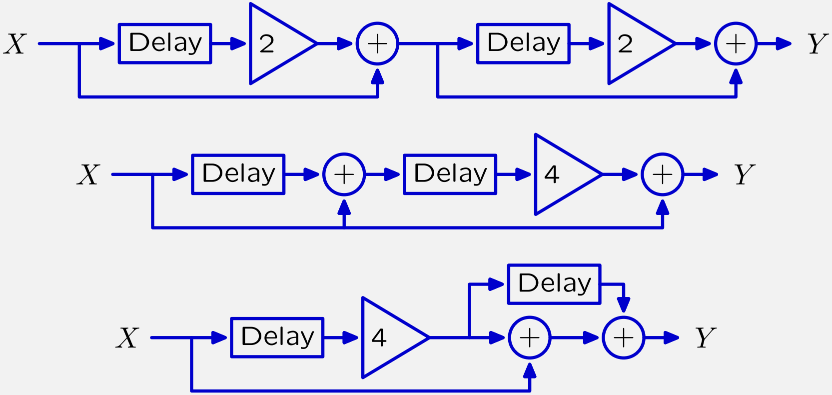
$$Y = (4\mathcal{R}(\mathcal{R}+1)+1)\,X$$

All implement $Y = (4\mathcal{R}^2+4\mathcal{R}+1)\,X$

Check Yourself

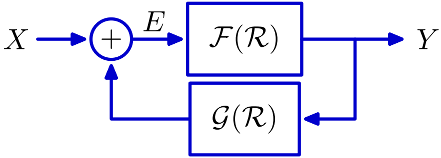
How many of the following systems are equivalent to

$$Y = (4\mathcal{R}^2 + 4\mathcal{R} + 1)\, X \quad ? \qquad 3$$



Using Operators to Analyze Feedback Systems

Simple feedback systems can contain both a forward path $\mathcal{F}(\mathcal{R})$ and a feedback path $\mathcal{G}(\mathcal{R})$.



$$Y = \mathcal{F}(\mathcal{R})E = \mathcal{F}(\mathcal{R})\Big(X + \mathcal{G}(\mathcal{R})Y\Big) = \mathcal{F}(\mathcal{R})X + \mathcal{F}(\mathcal{R})\mathcal{G}(\mathcal{R})Y$$

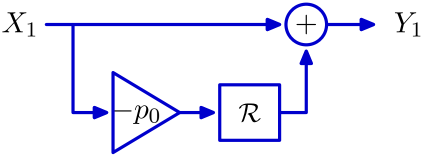
$$\Big(1 - \mathcal{F}(\mathcal{R})\mathcal{G}(\mathcal{R})\Big)Y = \mathcal{F}(\mathcal{R})X$$

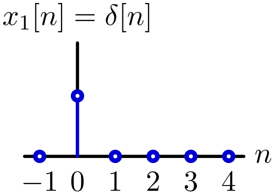
Feedback imposes an **implicit** relation between X and Y.

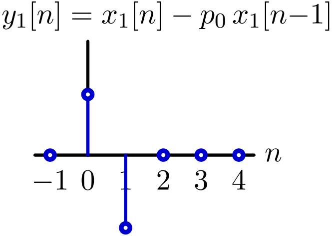
The output Y is the signal that produces $\mathcal{F}(\mathcal{R})$ when operated on by $(1 - \mathcal{F}(\mathcal{R})\mathcal{G}(\mathcal{R}))$.

Transient and Persistent Responses

The following system is feedforward. It has no cyclic signal-flow pathways. Consider its response to a "unit-sample signal" $\delta[n]$.







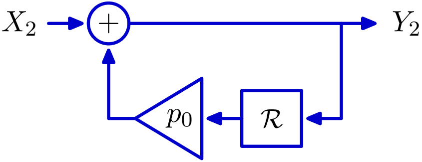
The duration of its response to a unit-sample signal is limited by the highest order term in its operator representation:

$$\mathcal{F}(\mathcal{R}) = 1 - p_0\mathcal{R}$$

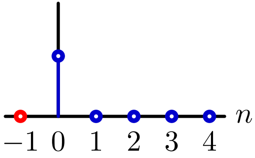
Transient and Persistent Responses

Systems with feedback can have **persistent** responses to **transient** inputs.

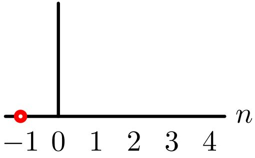
The following system has a cyclic signal-flow pathway.
Consider its response to a "unit-sample signal" $\delta[n]$.



$x_2[n] = \delta[n]$



$y_2[n] = x_2[n] + p_0\, y_2[n-1]$



Each cycle creates another sample in the output.

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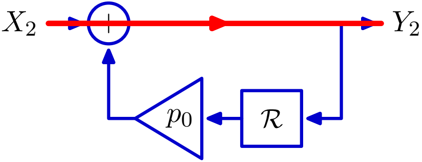

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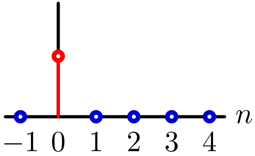

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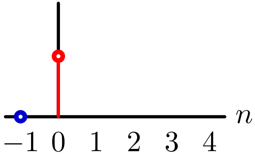

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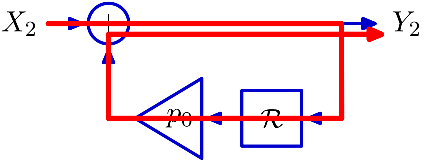

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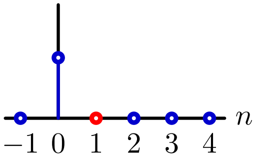

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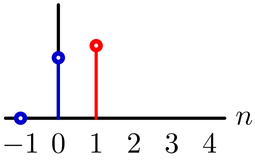

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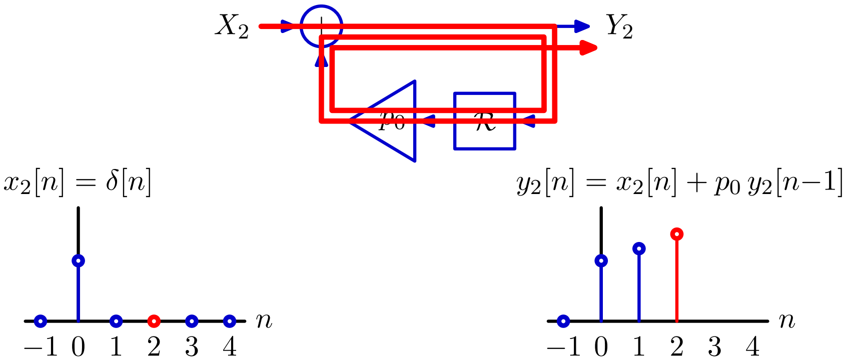

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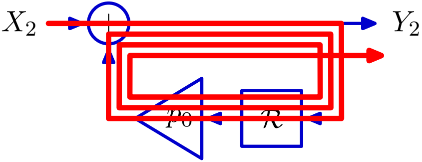
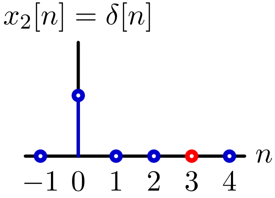
Transient and Persistent Responses

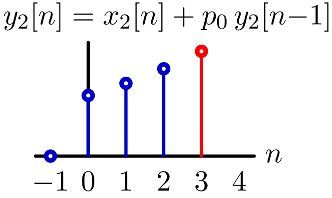
Systems with feedback can have **persistent** responses to **transient** inputs.

The following system has a cyclic signal-flow pathway.

Consider its response to a "unit-sample signal" $\delta[n]$.







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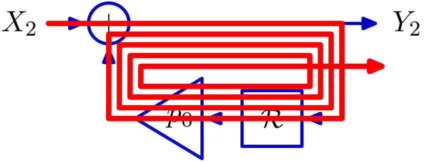
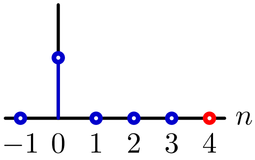
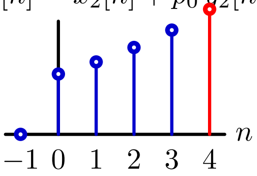


$x_2[n] = \delta[n]$



$y_2[n] = x_2[n] + p_0\, y_2[n-1]$



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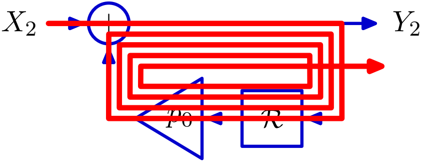
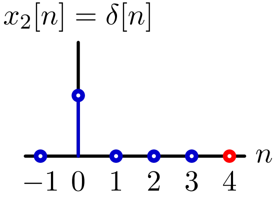
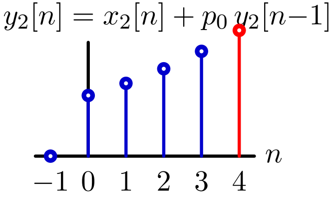






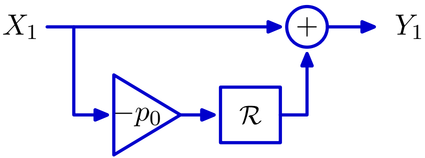
Each cycle creates another sample in the output.

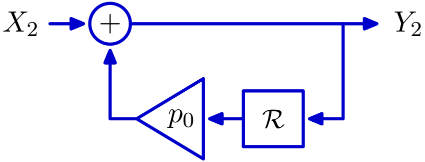
The output Y_2 persists forever even though the input $x_2[n] = 0$ for $n > 0$.

We say that this system has a **natural frequency** p_0.

Transient and Persistent Responses

Compare operator descriptions of these feedback and feedforward systems:





$$Y_1 = (1 - p_0\mathcal{R})X_1 \qquad\qquad (1 - p_0\mathcal{R})Y_2 = X_2$$

Transient and Persistent Responses

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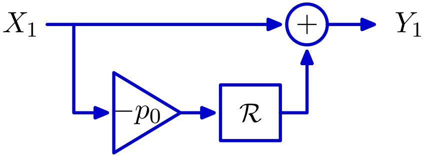

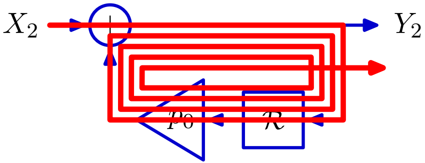

$$Y_1 = (1 - p_0\mathcal{R})X_1$$

$$(1 - p_0\mathcal{R})Y_2 = X_2$$

$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)X_2$$

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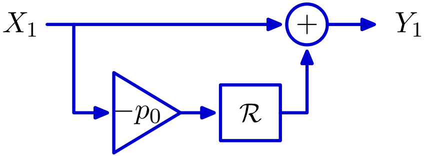

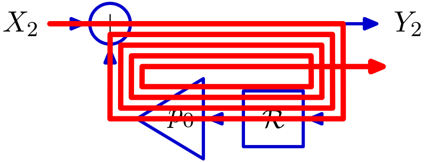

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$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)X_2$$

Substitute X_2 from the first equation into the second:

$$Y_2 = (1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)(1 - p_0\mathcal{R})Y_2$$

and therefore

$$(1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + \cdots)(1 - p_0\mathcal{R}) = 1$$

The two factors $1+p_0\mathcal{R}+p_0^2\mathcal{R}^2+p_0^3\mathcal{R}^3+\cdots$ and $1-p_0\mathcal{R}$ must be **reciprocals**.

We can think of the operator representation of this feedback system as

$$\mathcal{H}(\mathcal{R}) = \frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

Polynomial Interpretation of Reciprocals

The reciprocal relation between the two representations

$$\mathcal{H}(\mathcal{R}) = \frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \cdots$$

also follows from polynomial division.

$$\begin{array}{r|llllll}
 & 1 & +p_0\mathcal{R} & +p_0^2\mathcal{R}^2 & +p_0^3\mathcal{R}^3 & +\cdots & \\
\cline{2-6}
1 - p_0\mathcal{R} & 1 & & & & & \\
 & 1 & -p_0\mathcal{R} & & & & \\
\cline{2-3}
 & & p_0\mathcal{R} & & & & \\
 & & p_0\mathcal{R} & -p_0^2\mathcal{R}^2 & & & \\
\cline{3-4}
 & & & p_0^2\mathcal{R}^2 & & & \\
 & & & p_0^2\mathcal{R}^2 & -p_0^3\mathcal{R}^3 & & \\
\cline{4-5}
 & & & & p_0^3\mathcal{R}^3 & & \\
 & & & & p_0^3\mathcal{R}^3 & -p_0^4\mathcal{R}^4 & \\
\cline{5-6}
 & & & & & \cdots &
\end{array}$$

This is another instance of how the normal rules of polynomial algebra apply to system operators.

Summary

Today we introduced **polynomial** (aka operator) representations of discrete time systems.

→ polynomials are generally easier to work with than difference equations

→ polynomials provide insights not apparent from difference equations

Next time: Geometric Signals and System Functions