6.3100: Dynamic System Modeling and Control Design

System Functions, Poles, and Natural Frequencies

March 11, 2024

Retrospective

To date, we have used a number of different representations

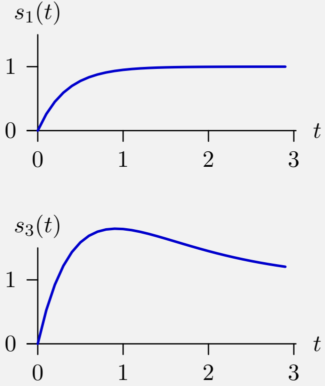
- differential and difference equations
- block diagrams
- system (transfer) functions
- poles and zeros

to get a variety of perspectives on the control of systems.

These are most useful when you can readily translate between them.

Check Yourself

Step responses of four systems that contain only adders, gains, differentiators, and integrators are shown below.



Which if any of the systems could have just one pole?

Which must have > 1 but no more than 2 poles (both real)?

System Functions, Poles, and Natural Frequencies

If a system contains only adders, gains, differentiators, and integrators, then its system function can be written as a **rational polynomial** in s:

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots}$$

The numerator and denominator polynomials can be factored into products of terms: one for each **zero** in the numerator or **pole** in the denominator:

$$H(s) = K \frac{(s - z_0)(s - z_1)(s - z_2) \cdots}{(s - p_0)(s - p_1)(s - p_2) \cdots}$$

Contributions of each pole can be **isolated** by expanding the system function in partial fractions:

$$H(s) = \frac{c_0}{s - p_0} + \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots$$

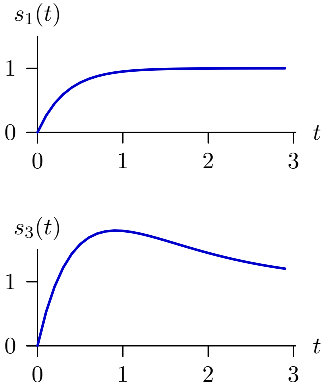
Each partial fraction term corresponds to a **first-order subsystem**:

$$H_i(s) = \frac{Y_i(s)}{X_i(s)} = \frac{c_i}{s - p_i}$$

with a **natural frequency** equal to the corresponding pole p_i.

Check Yourself

Only the first step response ($s_1(t)$) has an exponential shape that corresponds to a single pole.



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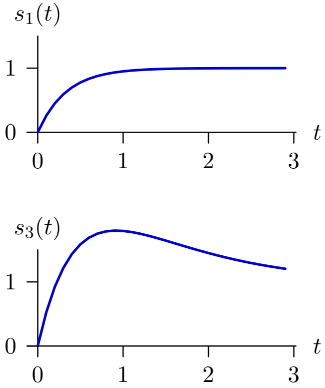
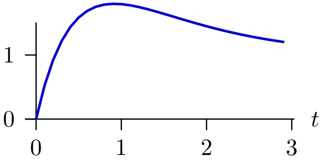
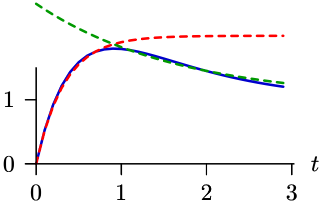


Which must have >1 but no more than 2 poles (both real)?

Sums of Exponentials

Although the third step response ($s_3(t)$) "overshoots" its final value, it is actually the sum of two exponentials.

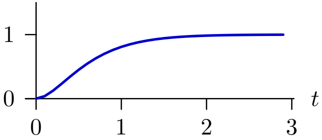




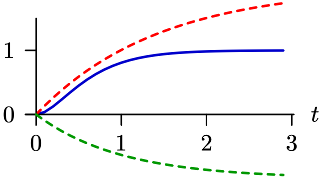
The faster pole contributes significantly to the fast initial rise of the step response and the slower pole contributes to the subsequent decline.

Sums of Exponentials

The second step response ($s_2(t)$) is also the sum of two exponentials.



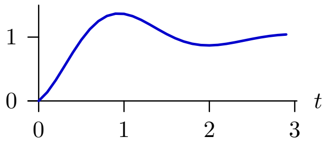
The two components have equal and opposite initial slopes, so the initial slope of the step response is zero – unlike the exponential response of $s_1(t)$.



The red component dominates the final value.

Complex Exponentials

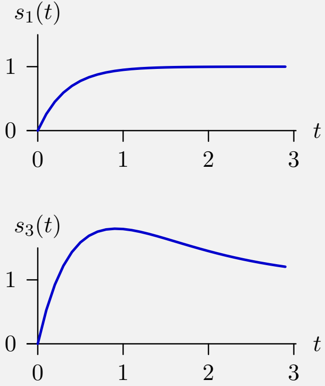
Only the fourth step response ($s_4(t)$) is oscillatory in the sense that the oscillations would persist forever if there were no damping in the system.



If the system has just two poles, they must have non-zero imaginary parts.

Check Yourself

Step responses of four systems that contain only adders, gains, differentiators, and integrators are shown below.

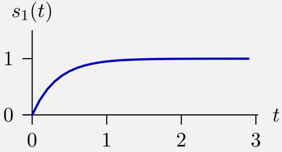


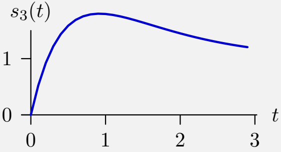
Which if any of the systems could have just one pole? **1**

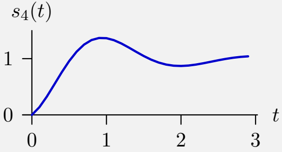
Which must have >1 but no more than 2 poles (both real)? **2,3**

Check Yourself

Match the step responses to their system functions below.

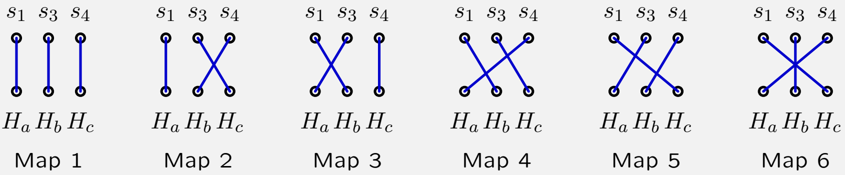






$$H_a(s) = \frac{6s+2}{s^2+3s+2} \qquad H_b(s) = \frac{s+10}{s^2+2s+10} \qquad H_c(s) = \frac{3}{s+3}$$

Which map shows the correct correspondence?



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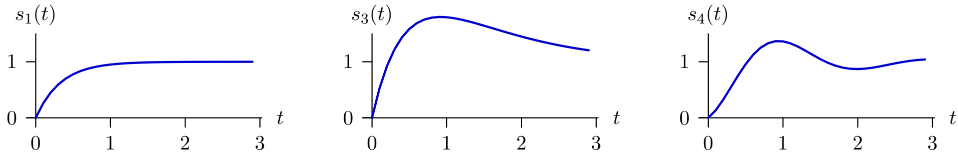


$$H_a(s) = \frac{6s+2}{s^2+3s+2} \qquad H_b(s) = \frac{s+10}{s^2+2s+10} \qquad H_c(s) = \frac{3}{s+3}$$

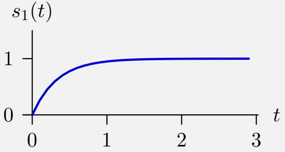
Only $s_1(t)$ can be written as a single exponential, and only $H_c(s)$ has a single pole. Therefore $H_c(s)$ corresponds to $s_1(t)$.

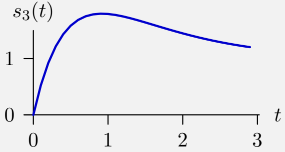
The denominator of $H_a(s)$ can be factored as $(s{+}1)(s{+}2)$.
The denominator of $H_b(s)$ can be factored as $(s{+}1{+}j3)(s{+}1{-}j3)$.

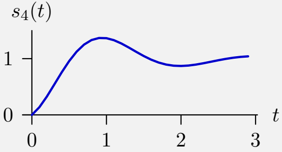
The complex poles of $H_b(s)$ correspond to the oscillations in $s_4(t)$.
The real poles of $H_a(s)$ correspond to the sum of two exponentials in $s_3(t)$.

Check Yourself

Match the step responses to their system functions below.

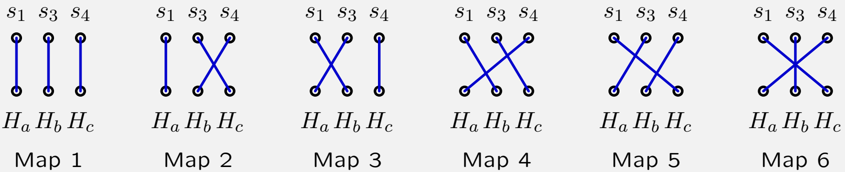






$$H_a(s) = \frac{6s+2}{s^2+3s+2} \qquad H_b(s) = \frac{s+10}{s^2+2s+10} \qquad H_c(s) = \frac{3}{s+3}$$

Which map shows the correct correspondence? Map 5



Complex-Valued Roots

Poles can have non-zero imaginary components (e.g., $H_b(s)$, previous slide).

$$H_b(s) = \frac{s+10}{s^2+2s+10} = \frac{s+10}{(s+1+j3)(s+1-j3)}$$

$$p = -1 \pm j3$$

Each pole is associated with a complex-valued eigenfunction: $e^{(-1\pm j3)t}$ but the step response of this system must have no imaginary part.

How is the step response real-valued when the poles have imaginary parts?

Complex-Valued Roots

The numerator and denominator polynomials of the system functions of systems the contain just adders, gains, differentiators, and integrators have real-valued coefficients.

If p is a root of a polynomial with constant real-valued coefficients, then its complex conjugate p^* is also a root.

Let $D(s)$ represent a polynomial in s with constant, real-valued coefficients.
If p is a root of $D(s)$ then $D(p) = 0$.[1]
Since all of the coefficients of $D(p)$ are real-valued,

$$D(p^*) = (D(p))^* = 0^* = 0\,.$$

Thus p^* is also a root.

---

1 That's the definition of a root.

Complex Roots

If we pair the factors corresponding to complex-conjugate roots, the resulting polynomial has real-valued coefficients.

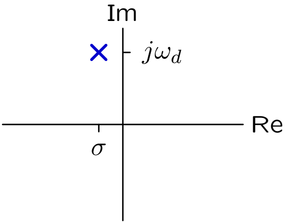
Let $p=\sigma+j\omega_0$. Then $p^*=\sigma-j\omega_0$.

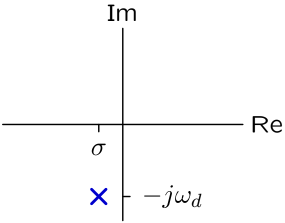
$$(s-p)(s-p^*)=s^2-2\,\mathrm{Re}\,(p)\,s+|p|^2$$

Both the real part of p and the squared magnitude of p are real-valued. Therefore the product of these factors has real-valued coefficients.

Natural Frequencies

Eigenfunctions of a system with a single complex pole are complex valued.





$$H_0(s) = \frac{1}{s - \sigma - j\omega_d} \qquad H_1(s) = \frac{1}{s - \sigma + j\omega_d}$$

$$h_0(t) = e^{(\sigma + j\omega_d)t} = e^{\sigma t}(\cos(\omega_d t) + j\sin(\omega_d t)) \quad h_1(t) = e^{(\sigma - j\omega_d)t} = e^{\sigma t}(\cos(\omega_d t) - j\sin(\omega_d t))$$

$$H_0(s) + H_1(s) = \frac{1}{s^2 - 2\sigma s + \sigma^2 + \omega_d^2}$$

$$h_0(t) + h_1(t) = 2e^{\sigma t}\cos(\omega_d t)$$

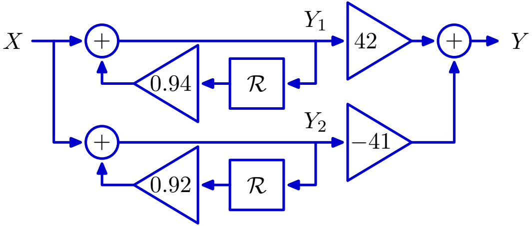
Responses of a system with complex poles in conjugate pairs are real-valued.

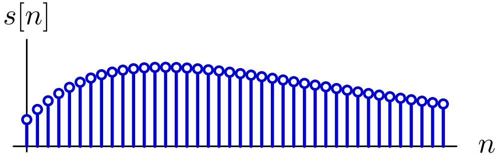
Comparison to DT

Comparison to DT

While the step response of a second order DT system with real-valued poles is a sum of geometrics, it may "overshoot" the final value.

$$\frac{Y}{X} = \frac{42}{1-0.94\mathcal{R}} - \frac{41}{1-0.92\mathcal{R}}$$





As in CT, this overshoot should not be confused with oscillations that occur with complex valued poles.

Complex Poles

What if a pole has a non-zero imaginary part?

Example:

$$\frac{Y}{X} = \frac{1}{1-\mathcal{R}+\mathcal{R}^2} = \frac{1}{1-\frac{1}{z}+\frac{1}{z^2}} = \frac{z^2}{z^2-z+1}$$

The poles are at $z=\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$, corresponding eigenfunctions are $(\frac{1}{2}\pm j\frac{\sqrt{3}}{2})^n$.

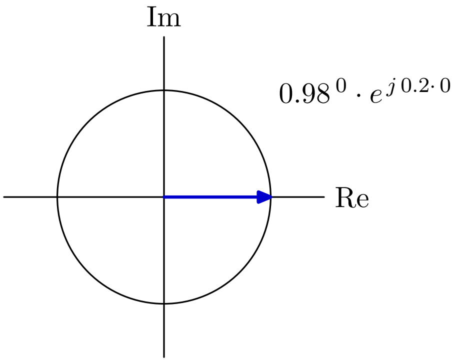
Powers of complex numbers are easy to compute using polar forms.
Express the pole at $z=a+jb$ as $re^{j\theta}$ where $r^2=a^2+b^2$ and $\tan\theta=\frac{b}{a}$.
Then the eigenfunction is $\left(re^{j\theta}\right)^n = r^n e^{j\theta n}$:

- geometric growth of magnitude
- linear growth of angle

Complex Pole Example

Consider a complex pole at $re^{j\theta}$ where $r = 0.98$ and $\theta = 0.2$.

The n^{th} sample of the corresponding eigenfunction is $(re^{j\theta})^n = r^n e^{j\theta n}$.



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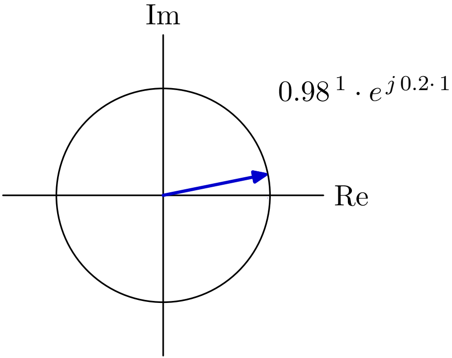

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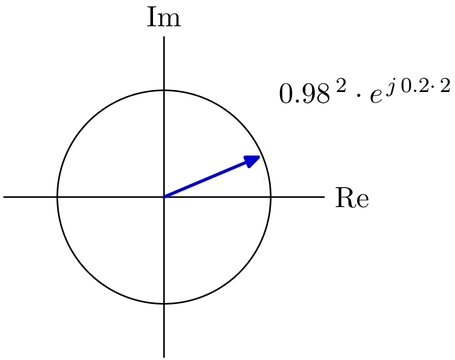

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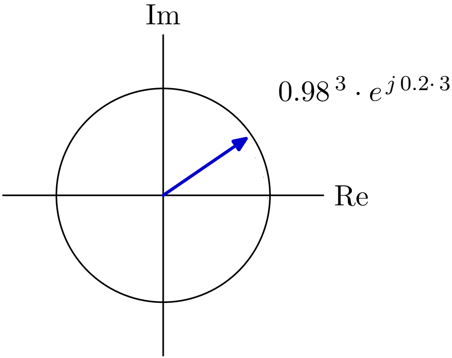

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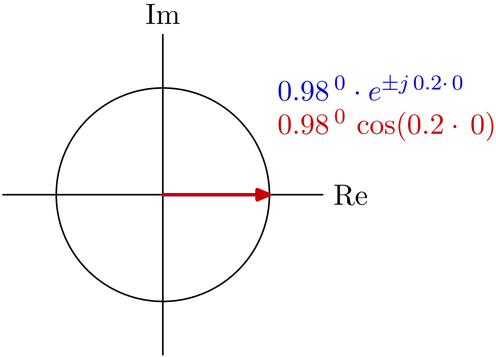
Complex Conjugates

The sum of the eigenfunctions associated with complex conjugates is real.

$$(re^{j\theta})^n + (re^{-j\theta})^n = r^n 2\cos\theta n$$

Example: $r = 0.98$, $\theta = 0.2$



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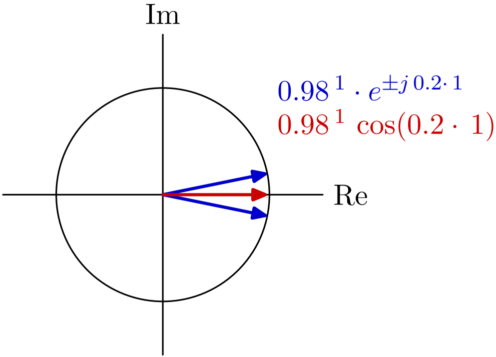

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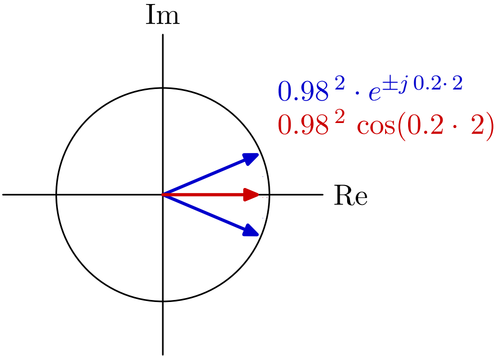

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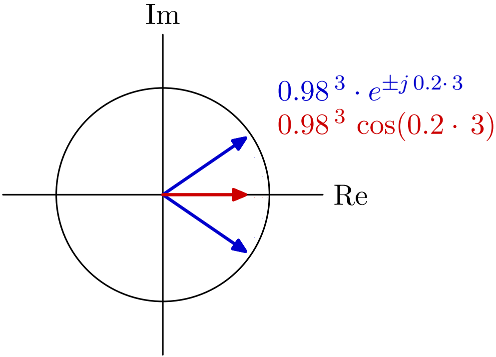
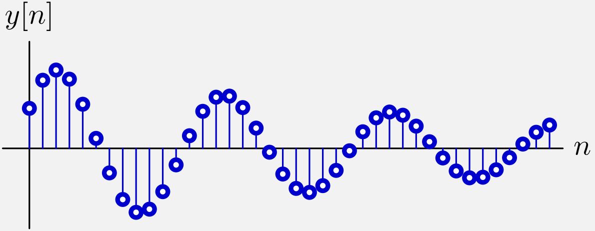


Check Yourself

Output of a system with poles at $z = re^{\pm j\Omega}$.

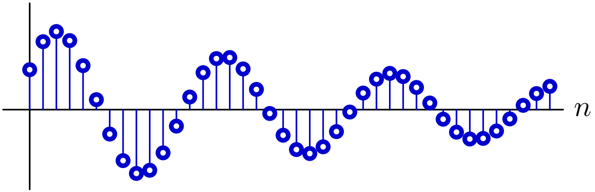


Which statement is true?

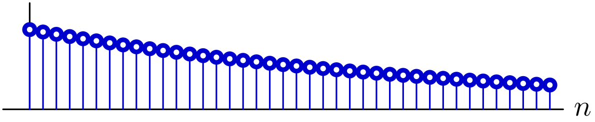
1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Check Yourself

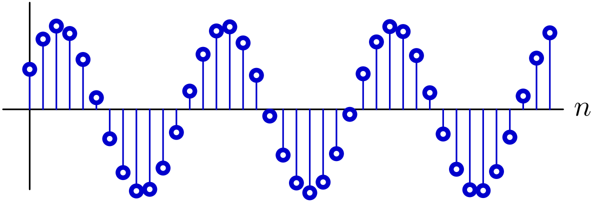
$$y[n] = Az_0^n + Bz_1^n = Ar^n e^{+jn\Omega} + Br^n e^{-jn\Omega}$$



$$r^n$$



$$Ae^{+jn\Omega} + Be^{-jn\Omega}$$



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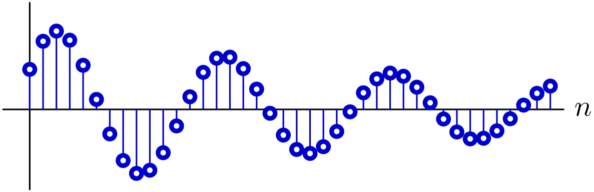
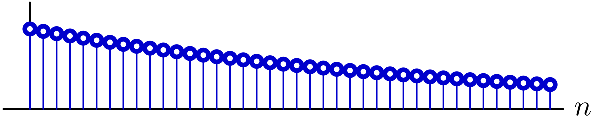
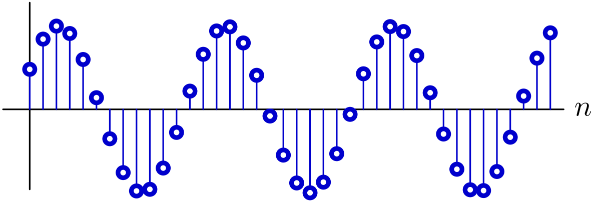


r^n



$r > 0.95$

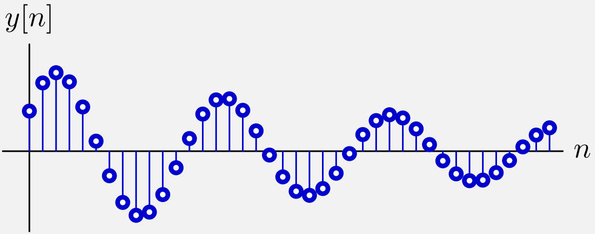
$Ae^{+jn\Omega} + Be^{-jn\Omega}$



period $\approx 12 \rightarrow \Omega \approx 0.5$

Check Yourself

Output of a system with poles at $z = re^{\pm j\Omega}$.



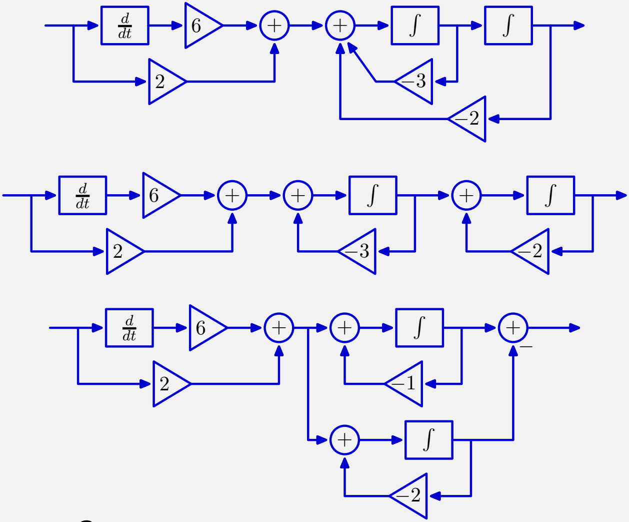
Which statement is true? 2

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Block Diagrams

Except for 1 error, these 3 block diagrams represent the same $H(s)$:

$$H(s) = \frac{6s+2}{s^2+3s+2}$$



What is the error?

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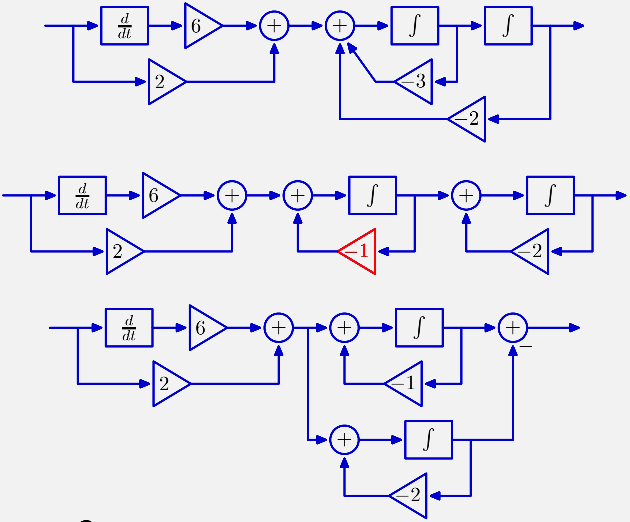

What is the error?

The gain of -3 near the center of middle figure should be -1.

Summary

Today we developed relations between a number of different representations os systems:

- differential and difference equations
- block diagrams
- system (transfer) functions
- poles and zeros

Next time we will add **frequency responses** to this list.