6.3100: Dynamic System Modeling and Control Design

Frequency Response and Bode Diagrams

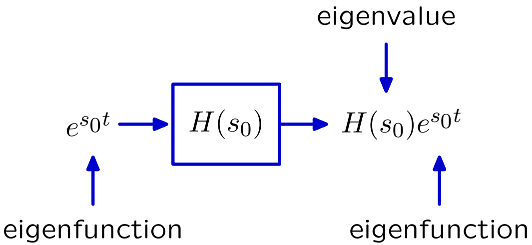
March 13, 2024

Retrospective: Eigenfunctions and Eigenvalues

If a system contains only adders, gains, differentiators, and integrators, then its system function can be written as a **rational polynomial** in s:

$$H(s) = \frac{Y}{X}$$

The eigenfunctions of such systems are **complex exponentials** ($e^{s_0 t}$) and associated eigenvalues are given by the system function evaluated at s_0.



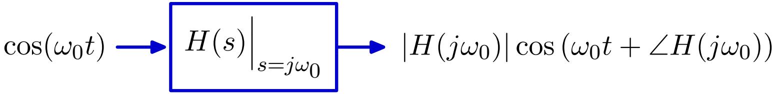
Retrospective: Frequency Response

The response of such a system to a sinusoidal input is determined by its system function $H(s)$ evaluated at $s{=}j\omega_0$.

$$e^{j\omega_0 t} \rightarrow H(j\omega_0)e^{j\omega_0 t}$$

$$e^{-j\omega_0 t} \rightarrow H(-j\omega_0)e^{-j\omega_0 t}$$

$$\begin{aligned}
\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) &\rightarrow \frac{1}{2}\left(H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right) \\
&\rightarrow \mathrm{Re}\left(H(j\omega_0)e^{j\omega_0 t}\right) \\
&\rightarrow \mathrm{Re}\left(|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t}\right) \\
&\rightarrow |H(j\omega_0)|\,\mathrm{Re}\left(e^{j\angle H(j\omega_0)+j\omega_0 t}\right) \\
&\rightarrow |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))
\end{aligned}$$



Retrospective: Poles and Zeros

If a system contains only adders, gains, differentiators, and integrators, then its system function can be written as a **rational polynomial** in s:

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots}$$

The numerator and denominator polynomials can be factored into products of terms: one for each **zero** in the numerator or **pole** in the denominator:

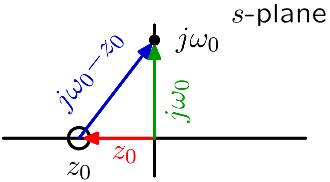
$$H(s) = K \frac{(s - z_0)(s - z_1)(s - z_2) \cdots}{(s - p_0)(s - p_1)(s - p_2) \cdots}$$

Even a complicated system is completely described by a small number of constants: K, z_0, p_0, z_1, p_1, z_2, p_2,...

Vector Interpretation of the Frequency Response

The frequency response of a system is determined by the ratio of the product of vectors from each zero to the point $j\omega_0$ divided by the product of vectors from each pole to the point $j\omega_0$.

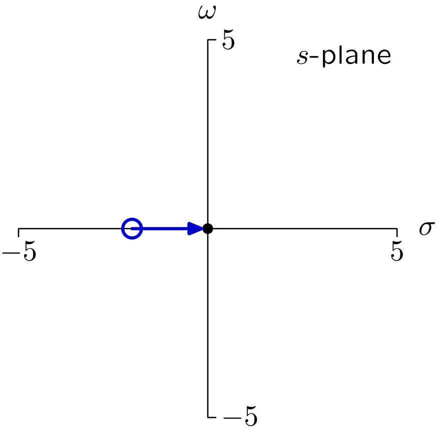
$$H(j\omega_0) = K\frac{(j\omega_0 - z_0)(j\omega_0 - z_1)(j\omega_0 - z_2)\cdots}{(j\omega_0 - p_0)(j\omega_0 - p_1)(j\omega_0 - p_2)\cdots}$$

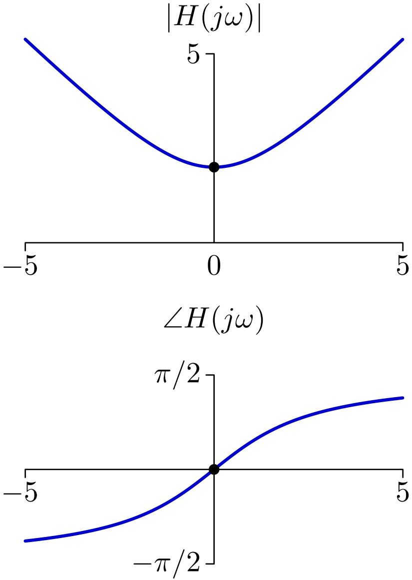


While this approach is simple (compared to solving a differential equation), using **Bode Diagrams** is even simpler.

Frequency Response: Single Zero

$$H(s) = s - z_1$$





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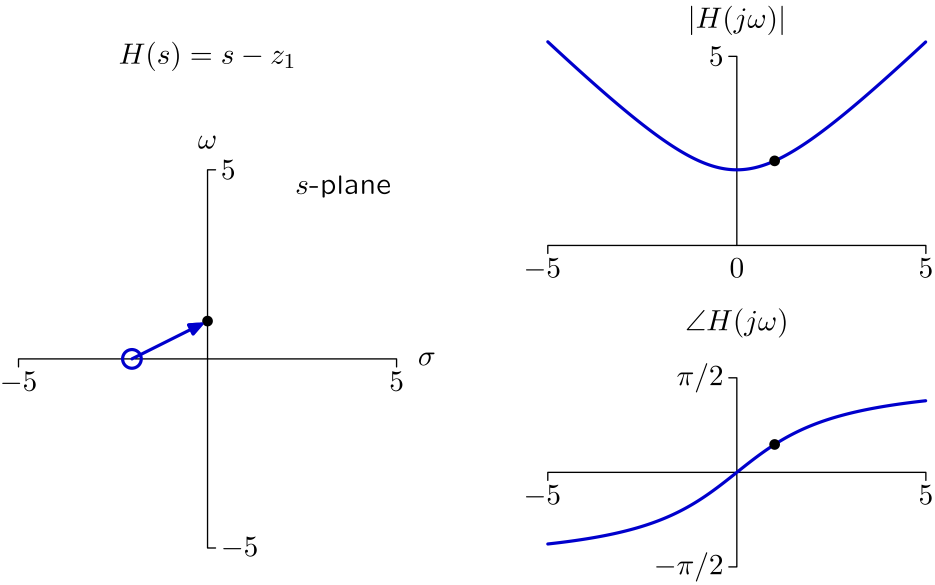

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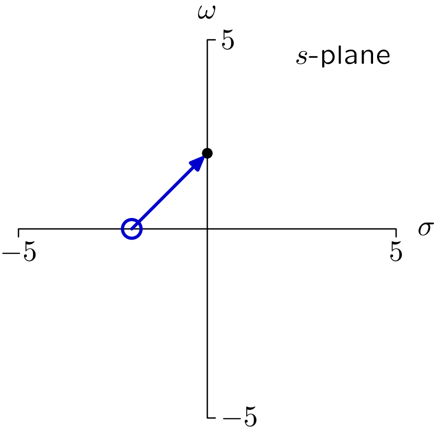

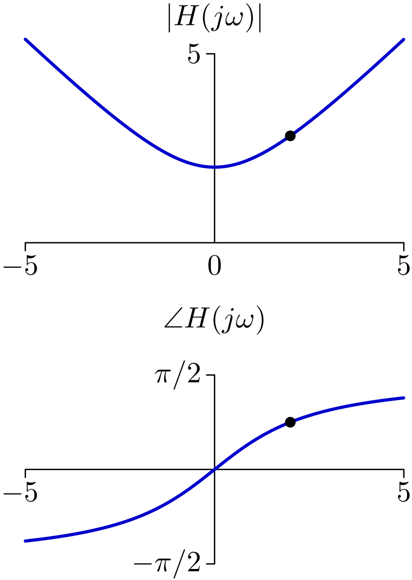

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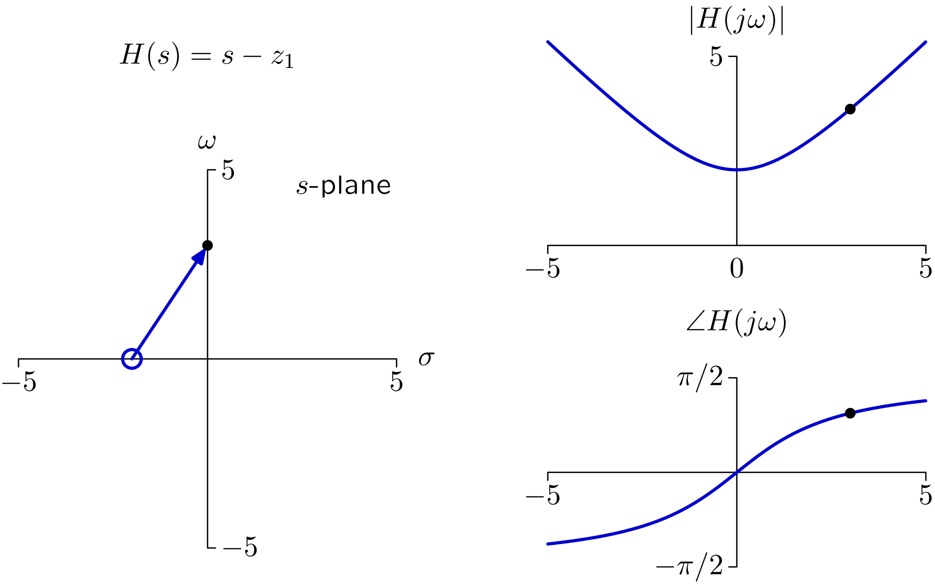

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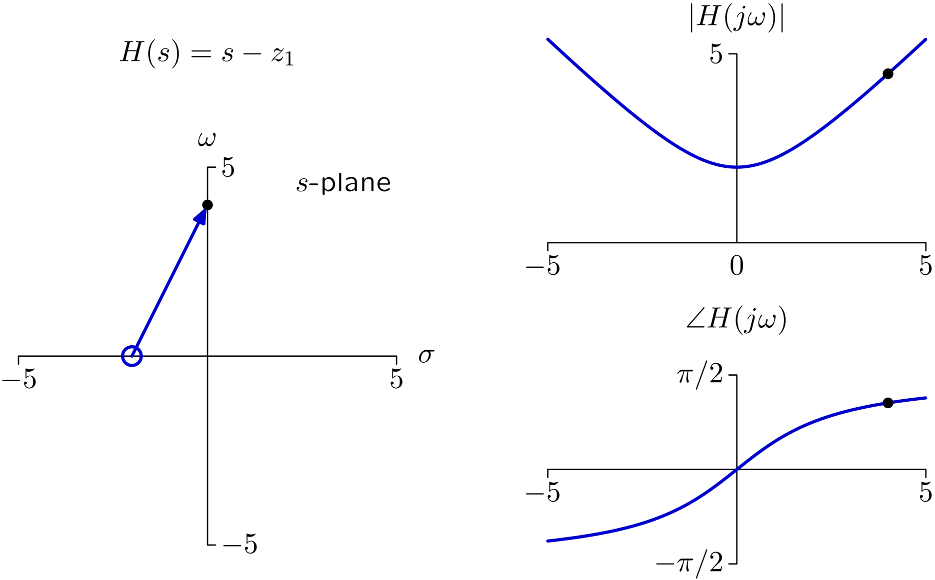

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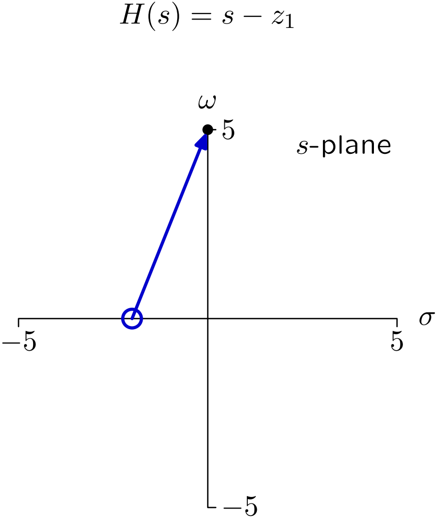
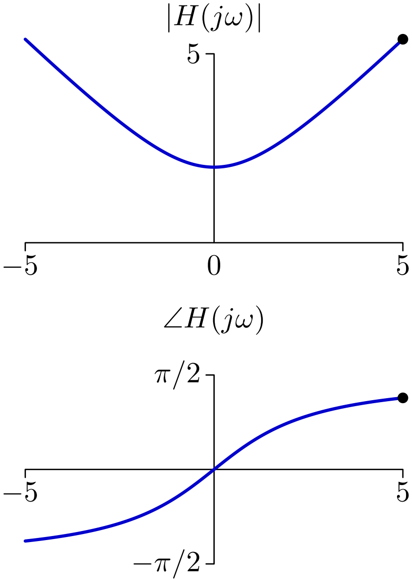
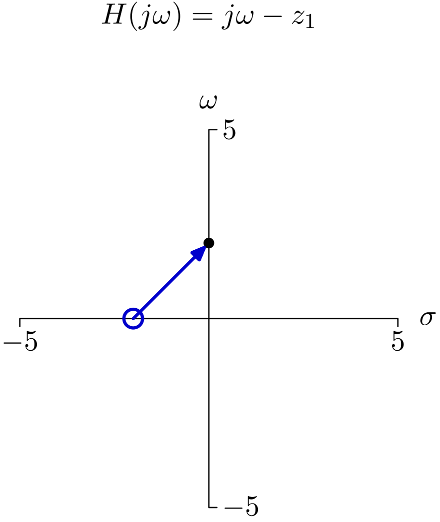
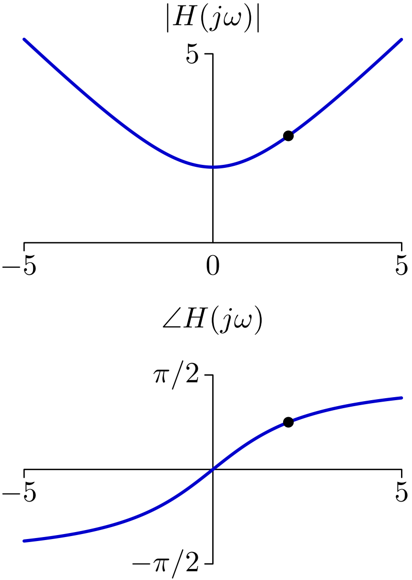




Asymptotic Behavior: Single Zero

The magnitude response is simple at low and high frequencies.





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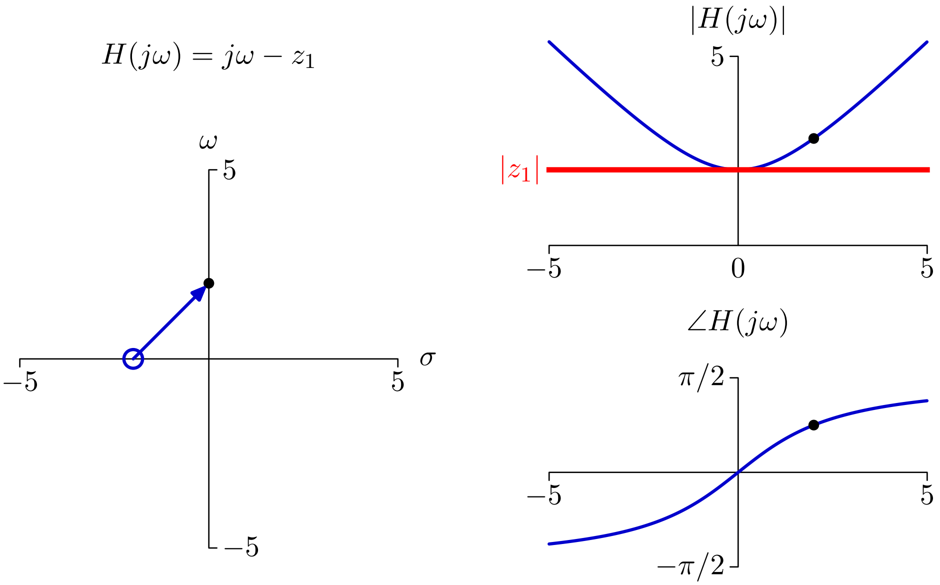

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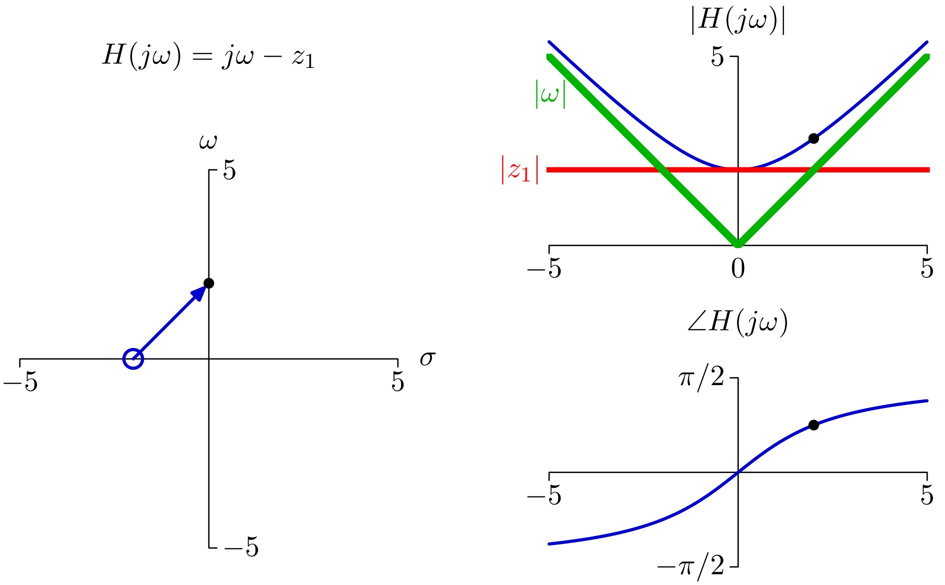

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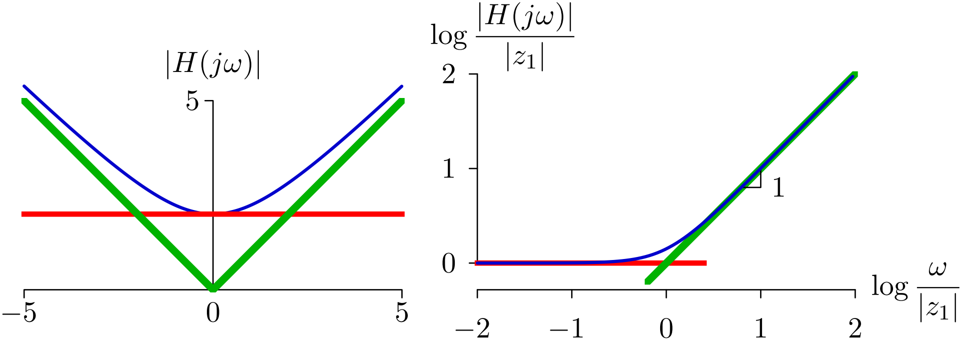
Two asymptotes provide a good approximation on log-log axes.

$$H(s) = s - z_1$$

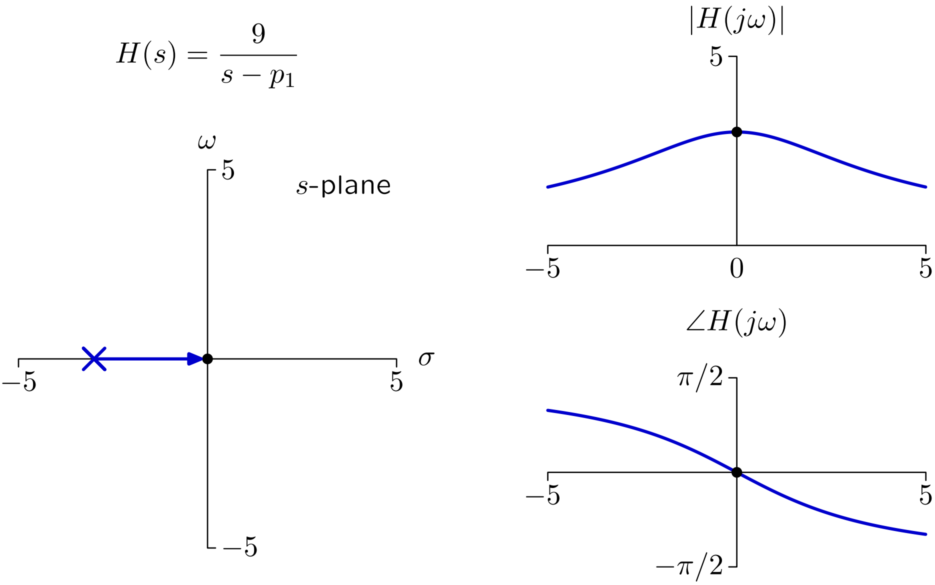


$$\lim_{\omega \to 0} |H(j\omega)| = |z_1|$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \omega$$

Frequency Response: Single Pole

$$H(s) = \frac{9}{s - p_1}$$



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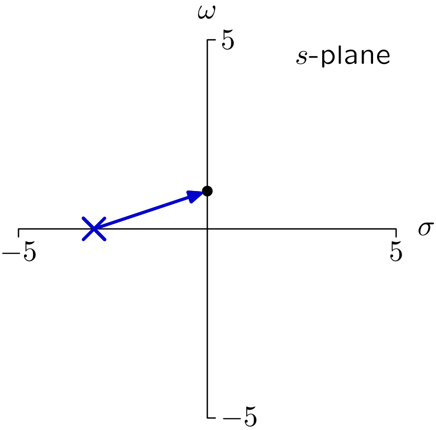

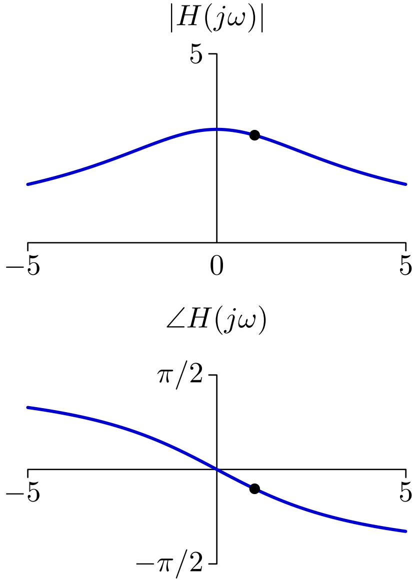

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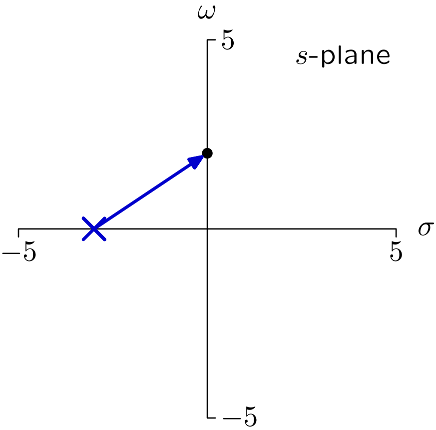

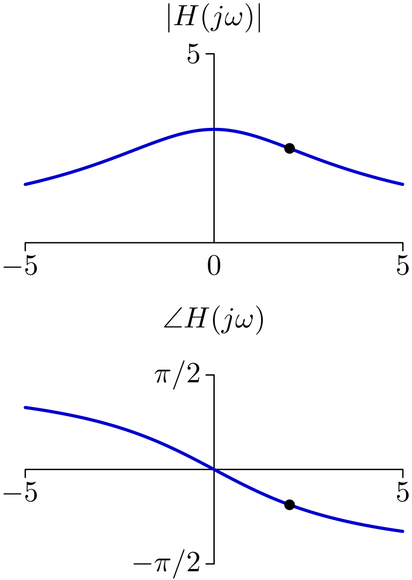

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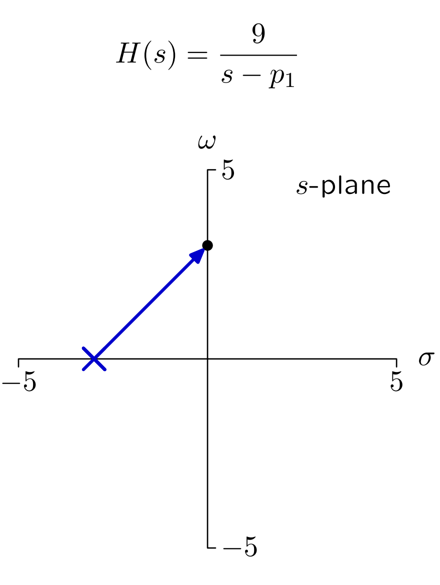

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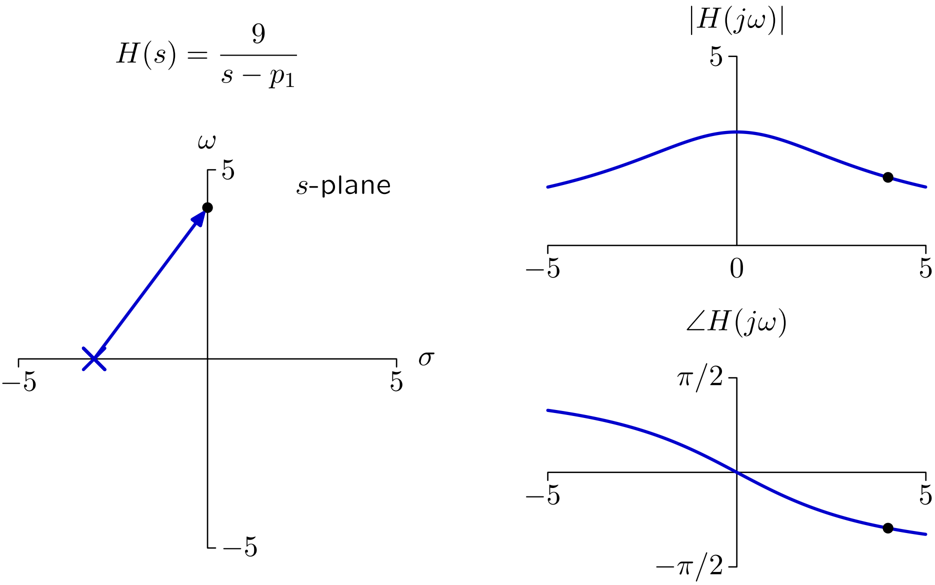

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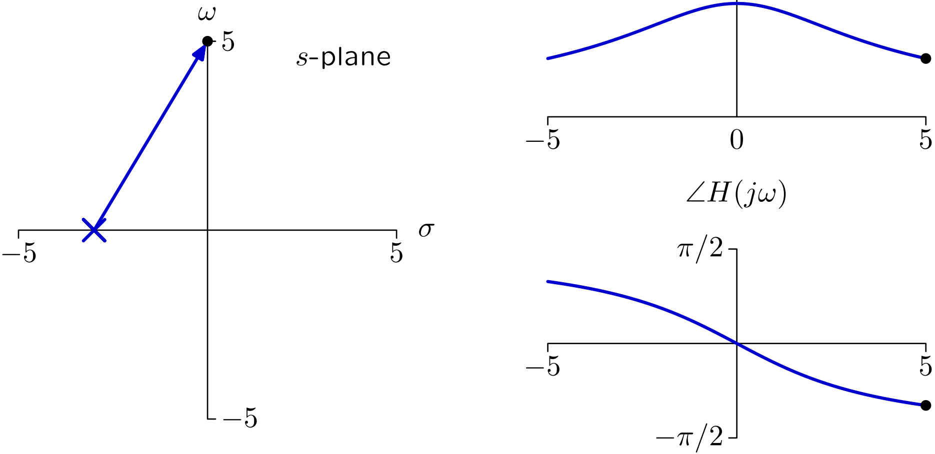

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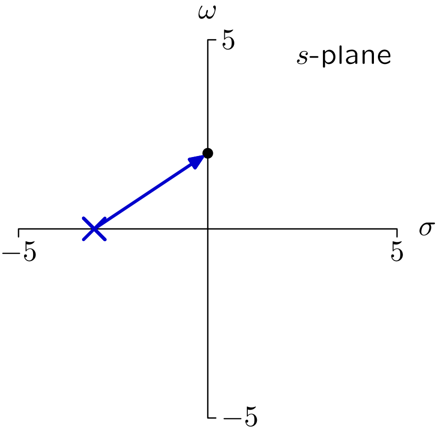

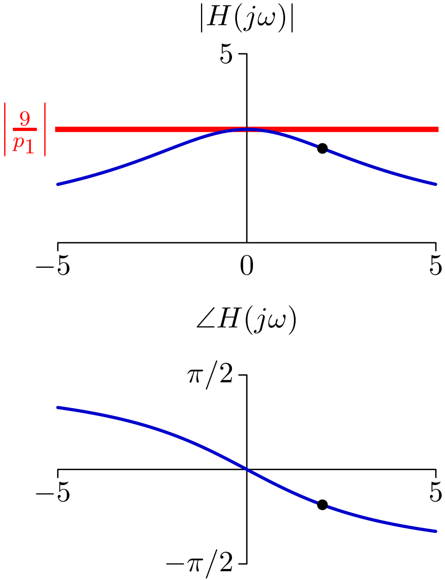

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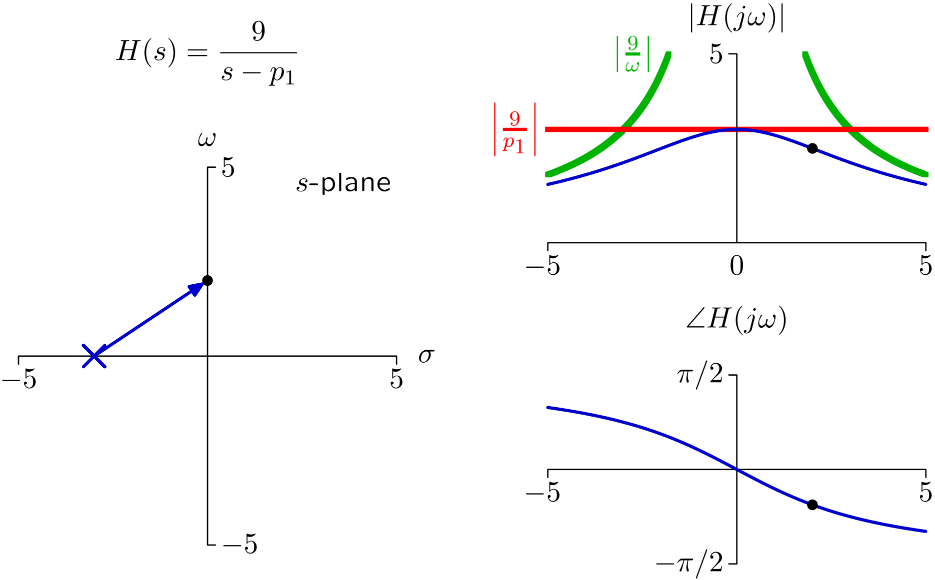

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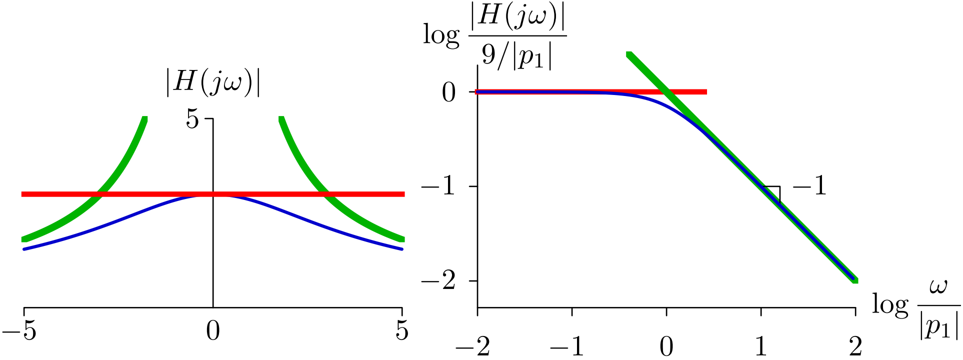

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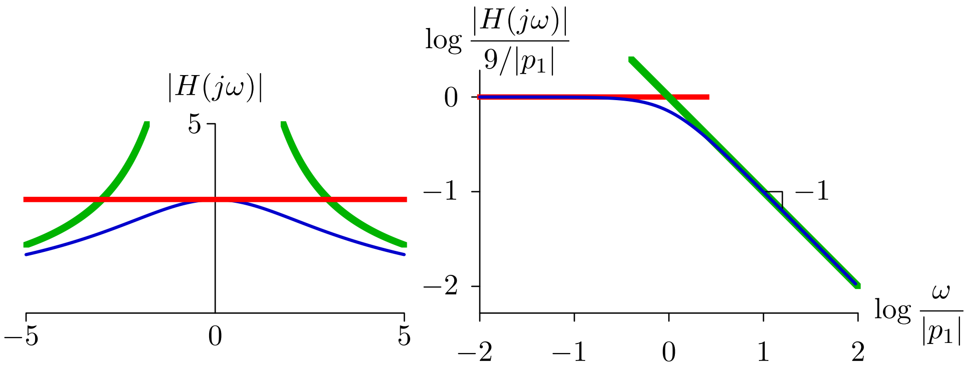

$$\lim_{\omega \to 0} |H(j\omega)| = \frac{9}{|p_1|}$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \frac{9}{\omega}$$

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \qquad \text{and} \qquad H_2(s) = \frac{1}{s+10}$$

The former can be transformed into the latter by

1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

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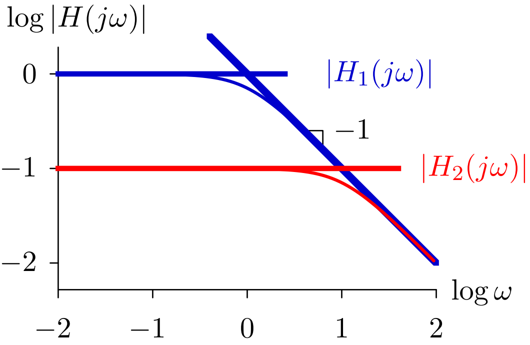

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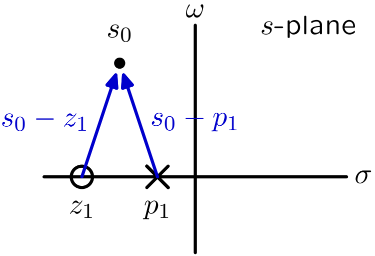
1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

no scaling in either vertical or horizontal directions !

Asymptotic Behavior of More Complicated Systems

Constructing $H(s_0)$.

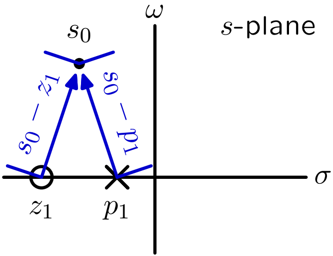
$$H(s_0) = K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \quad \begin{array}{l} \leftarrow \text{ product of vectors for zeros} \\ \\ \leftarrow \text{ product of vectors for poles} \end{array}$$



Asymptotic Behavior of More Complicated Systems

The magnitude of a product is the product of the magnitudes.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q}|s_0 - z_q|}{\prod_{p=1}^{P}|s_0 - p_p|}$$



Bode Plot

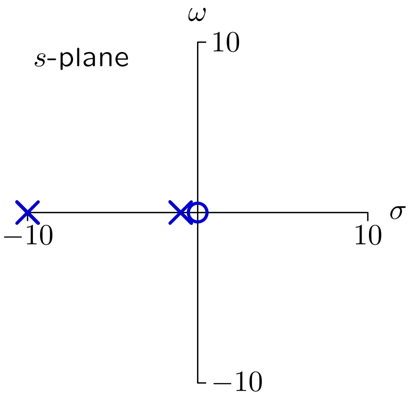
The log of the magnitude is a sum of logs.

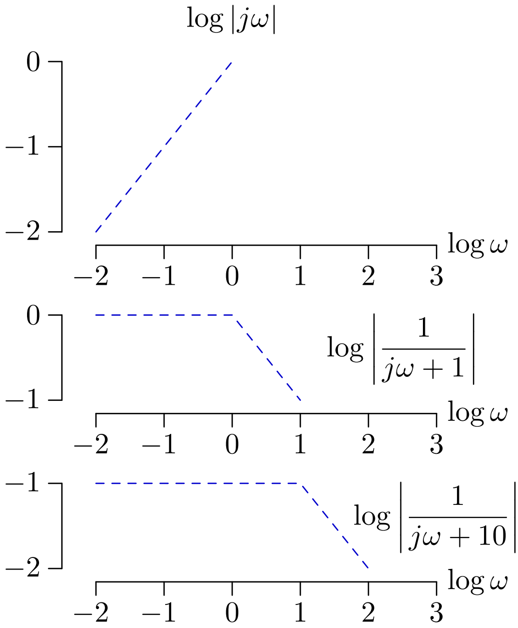
$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q} (s_0 - z_q)}{\prod_{p=1}^{P} (s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q} |s_0 - z_q|}{\prod_{p=1}^{P} |s_0 - p_p|}$$

$$\log |H(j\omega)| = \log |K| + \sum_{q=1}^{Q} \log |j\omega - z_q| - \sum_{p=1}^{P} \log |j\omega - p_p|$$

Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$





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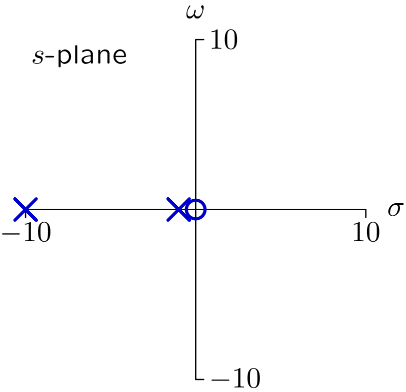

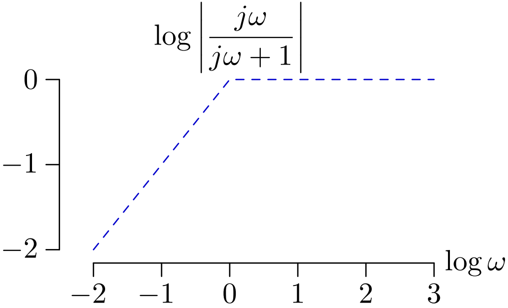

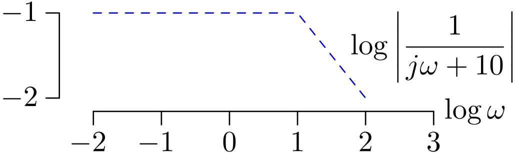

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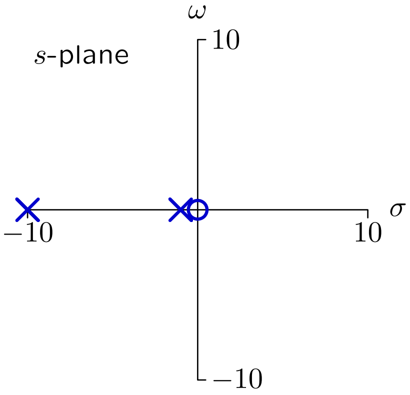

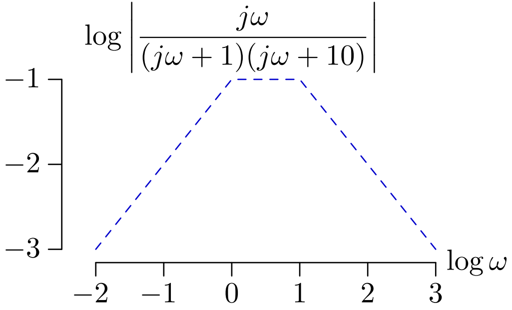

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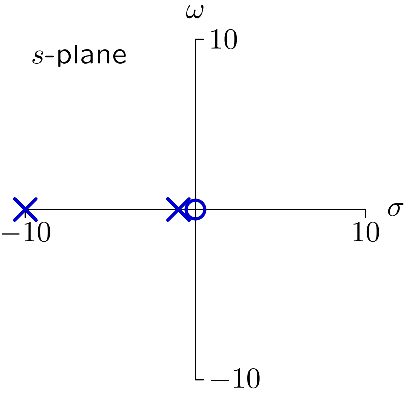
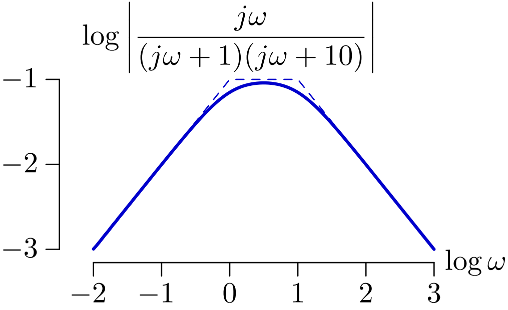
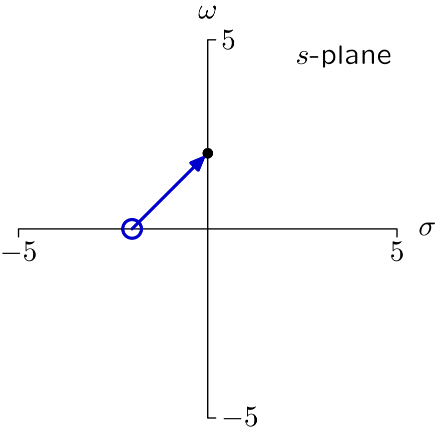


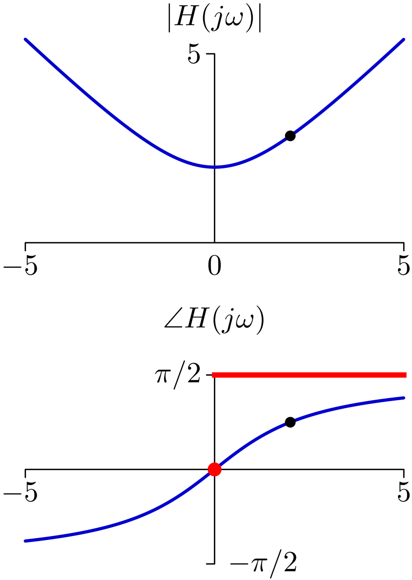


Asymptotic Behavior: Single Zero

The angle response is simple at low and high frequencies.

$$H(s) = s - z_1$$





Asymptotic Behavior: Single Zero

Three straight lines provide a good approximation versus log ω.

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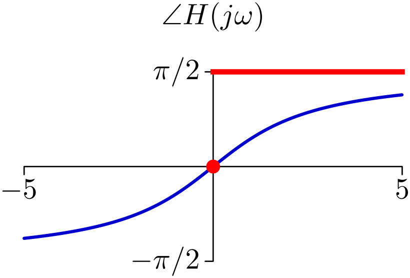
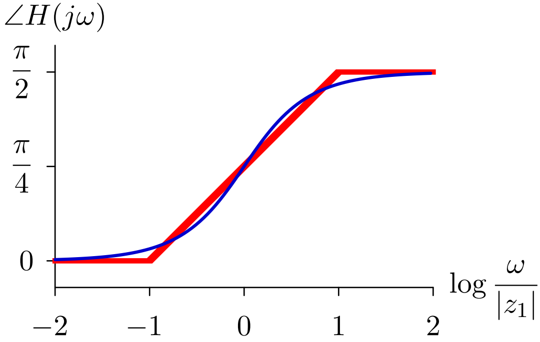




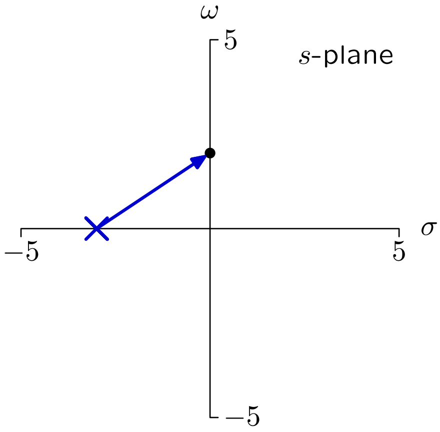
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

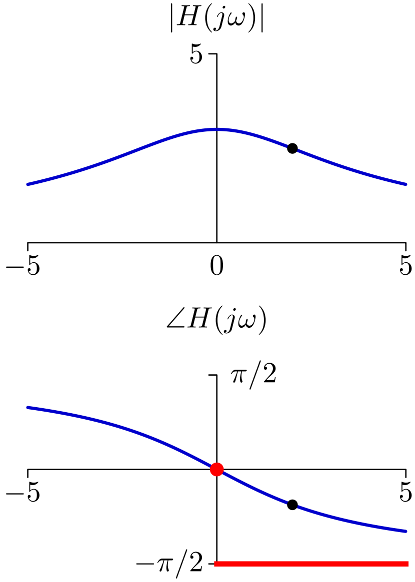
$$\lim_{\omega \to \infty} \angle H(j\omega) = \pi/2$$

Asymptotic Behavior: Single Pole

The angle response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$





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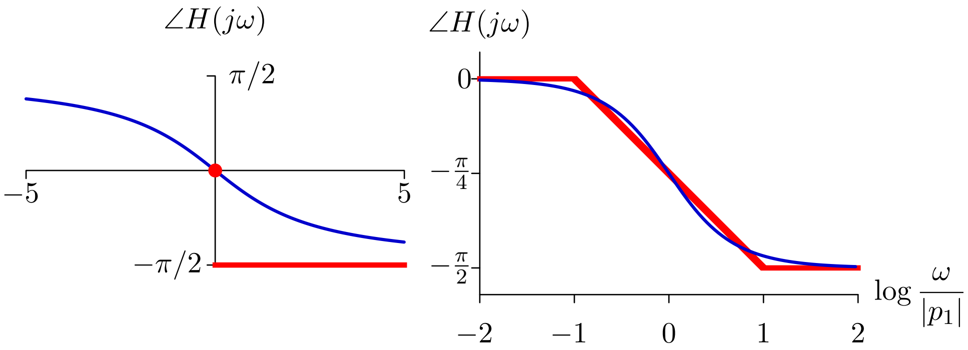


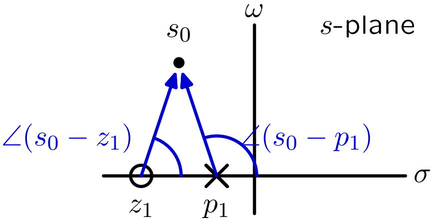
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = -\pi/2$$

Bode Plot

The angle of a product is the sum of the angles.

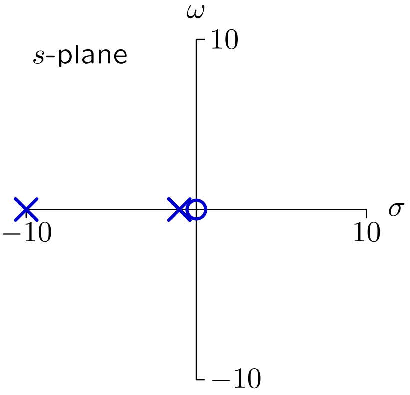
$$\angle H(s_0) = \angle \left(K \frac{\prod_{q=1}^{Q} (s_0 - z_q)}{\prod_{p=1}^{P} (s_0 - p_p)} \right) = \angle K + \sum_{q=1}^{Q} \angle (s_0 - z_q) - \sum_{p=1}^{P} \angle (s_0 - p_p)$$

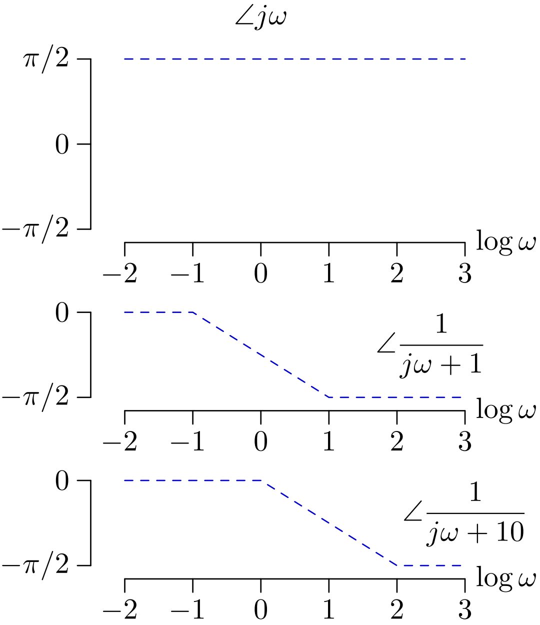


The angle of K can be 0 or π.

Bode Plot

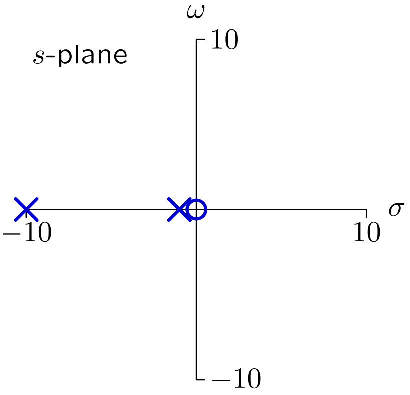
$$H(s) = \frac{s}{(s+1)(s+10)}$$

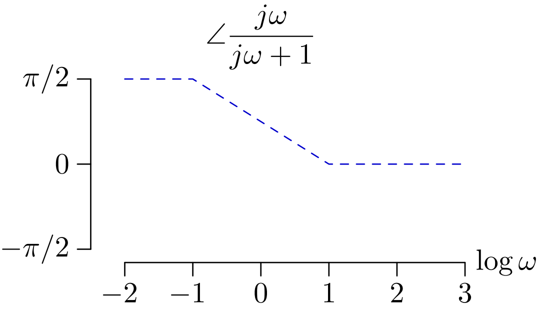


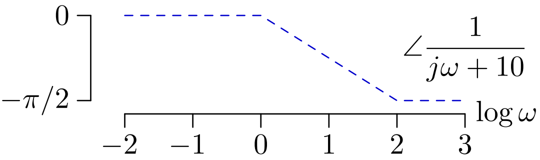


Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$

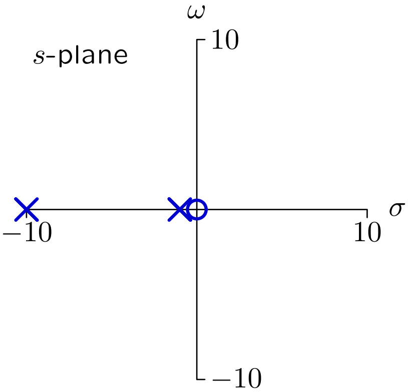


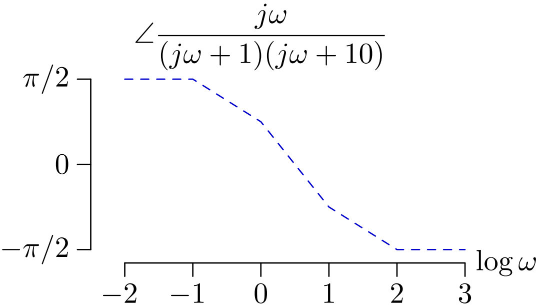




Bode Plot

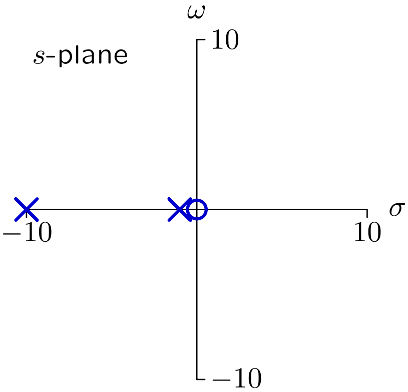
$$H(s) = \frac{s}{(s+1)(s+10)}$$

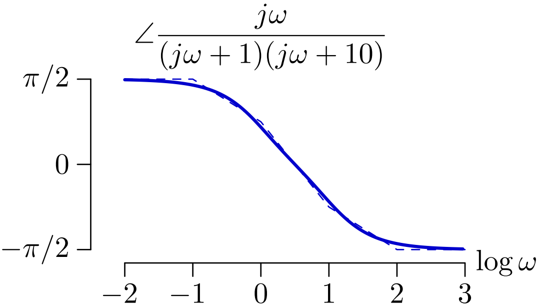




Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$





From Frequency Response to Bode Plot

The magnitude of $H(j\omega)$ is a product of magnitudes.

$$|H(j\omega)| = |K| \frac{\prod_{q=1}^{Q} |j\omega - z_q|}{\prod_{p=1}^{P} |j\omega - p_p|}$$

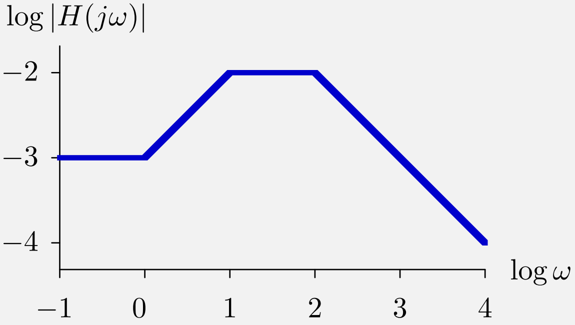
The log of the magnitude is a sum of logs.

$$\log|H(j\omega)| = \log|K| + \sum_{q=1}^{Q} \log|j\omega - z_q| - \sum_{p=1}^{P} \log|j\omega - p_p|$$

The angle of $H(j\omega)$ is a sum of angles.

$$\angle H(j\omega) = \angle K + \sum_{q=1}^{Q} \angle(j\omega - z_q) - \sum_{p=1}^{P} \angle(j\omega - p_p)$$

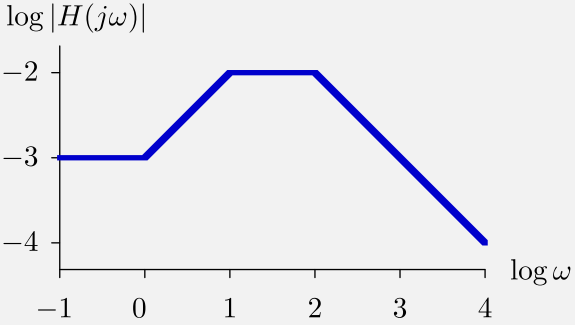
Check Yourself



Which corresponds to the Bode approximation above?

1. $\dfrac{1}{(s+1)(s+10)(s+100)}$
2. $\dfrac{s+1}{(s+10)(s+100)}$
3. $\dfrac{(s+10)(s+100)}{s+1}$
4. $\dfrac{s+100}{(s+1)(s+10)}$
5. none of the above

Check Yourself

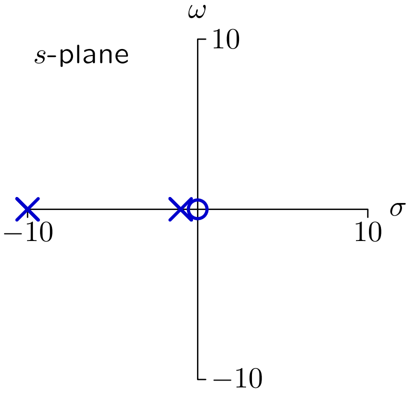


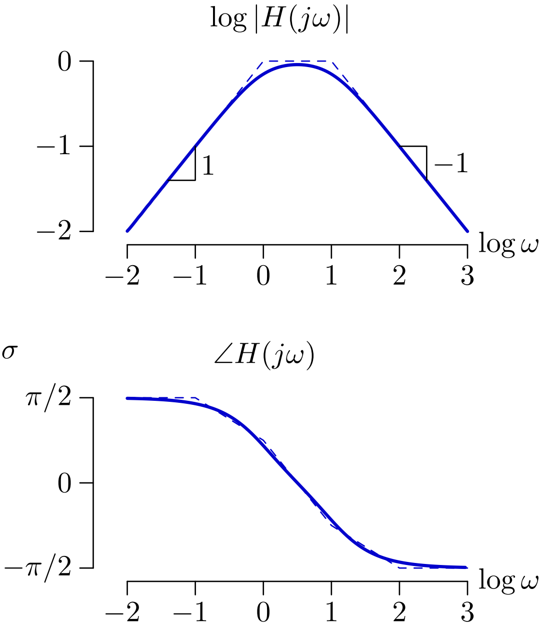
Which corresponds to the Bode approximation above? 2

1. $\dfrac{1}{(s+1)(s+10)(s+100)}$
2. $\dfrac{s+1}{(s+10)(s+100)}$
3. $\dfrac{(s+10)(s+100)}{s+1}$
4. $\dfrac{s+100}{(s+1)(s+10)}$
5. none of the above

Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$





Bode Plot: dB

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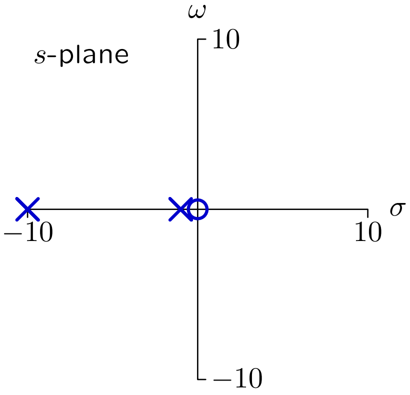
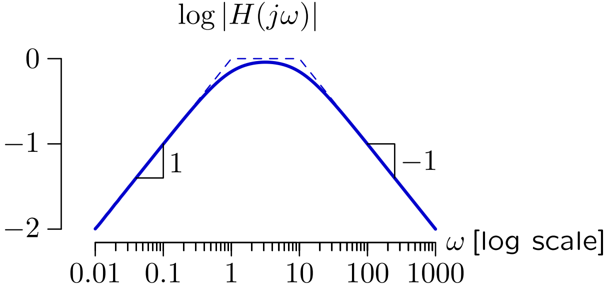
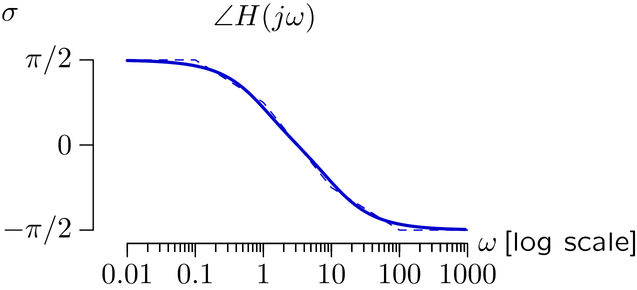
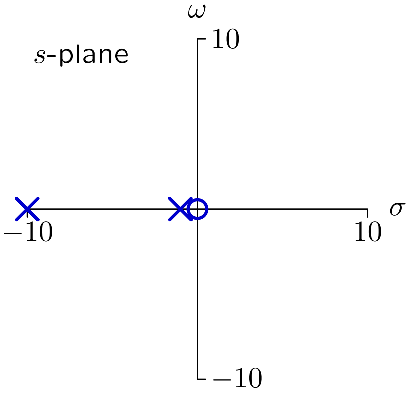


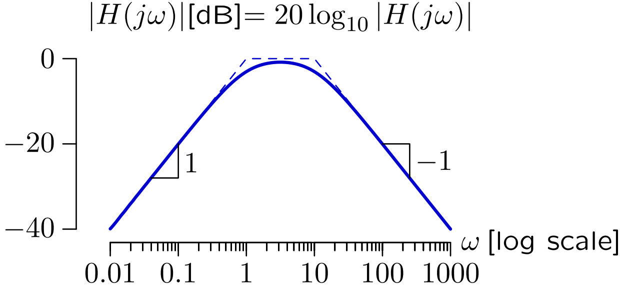


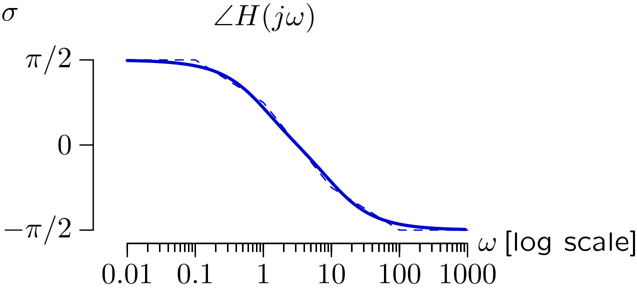


Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$

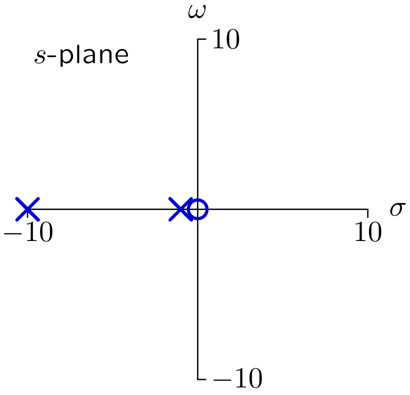


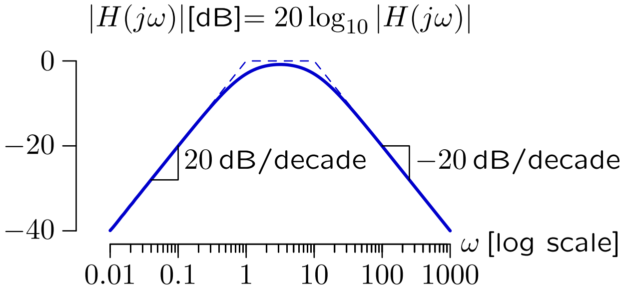


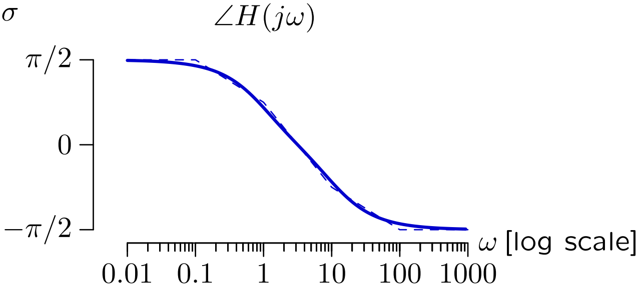


Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$



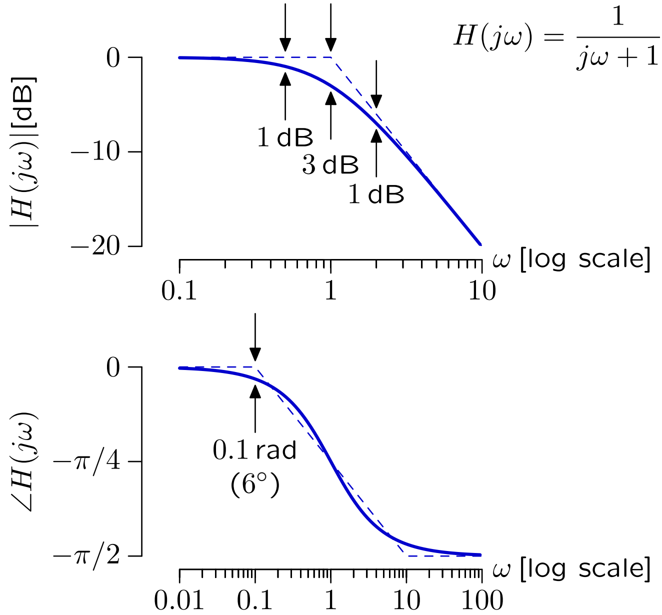




Bode Plot: Accuracy

The straight-line approximations are surprisingly accurate.

$$H(j\omega) = \frac{1}{j\omega + 1}$$



X	$20\log_{10} X$
1	0 dB
$\sqrt{2}$	≈ 3 dB
2	≈ 6 dB
10	20 dB
100	40 dB

Complex-Valued Poles and Zeros

New issues arise for complex-valued poles and zeros.

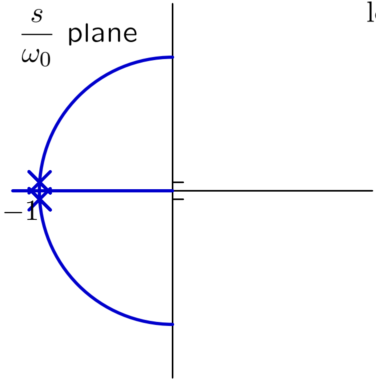
We have previously seen that complex-valued poles are associated with **resonance**. How does resonance affect a Bode plot?

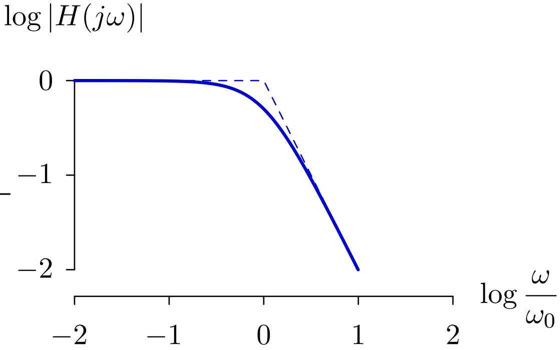
Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 0.501$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



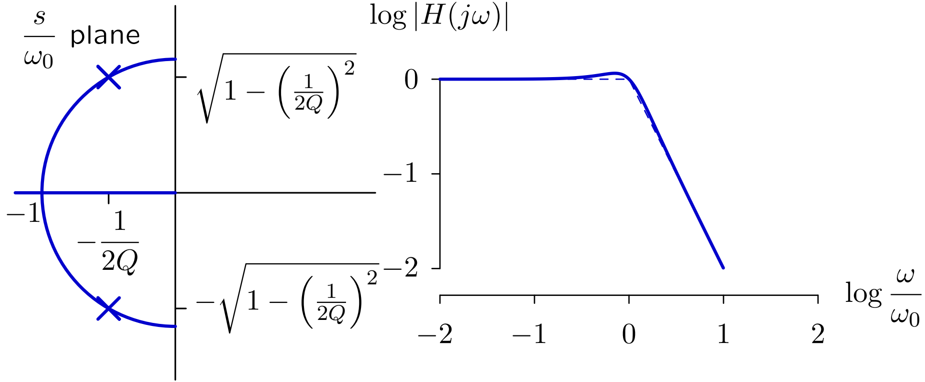


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$Q = 1$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

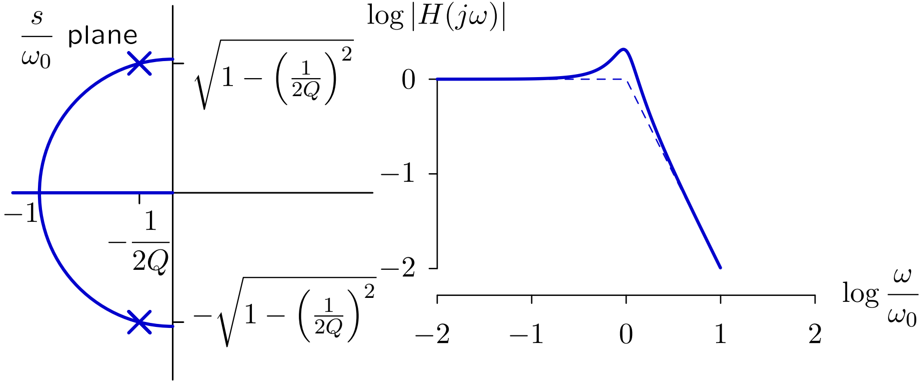


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 2$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

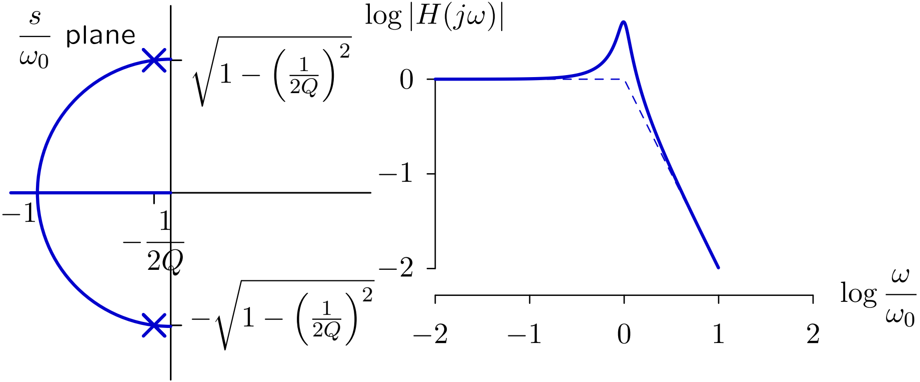


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 4$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

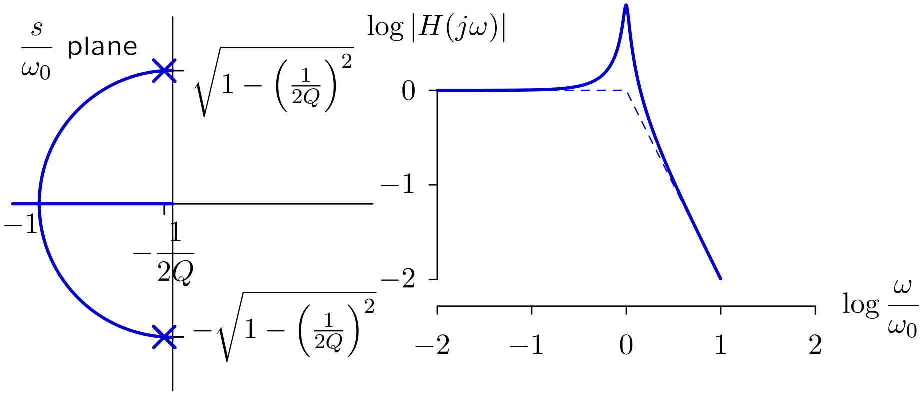


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 8$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

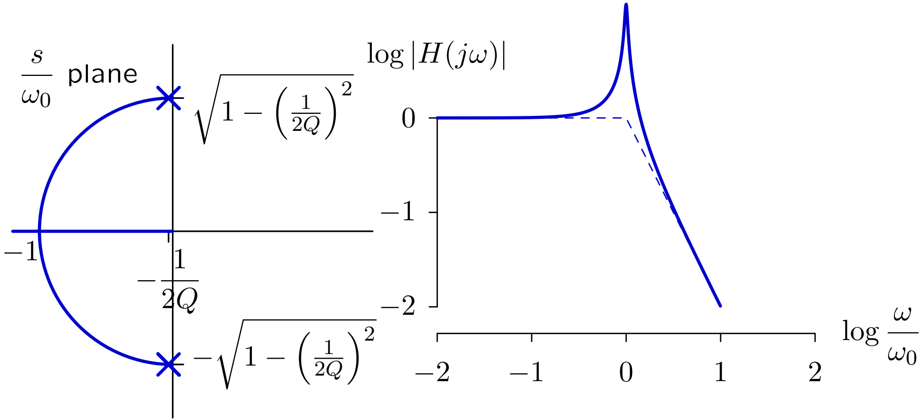


Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$Q = 16$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

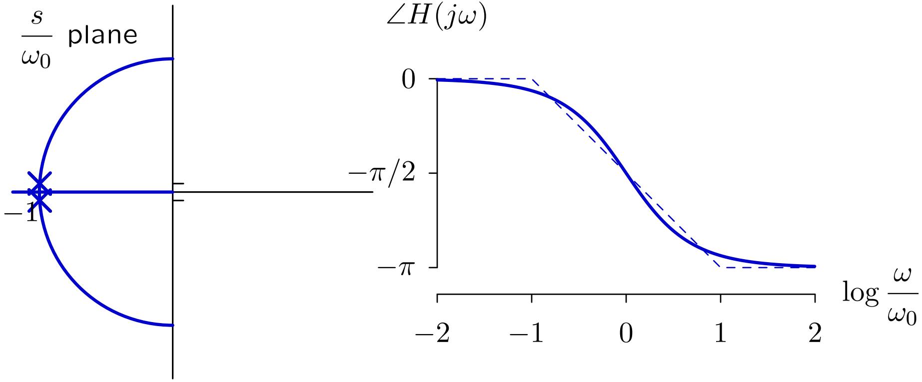


Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$Q = 0.501$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

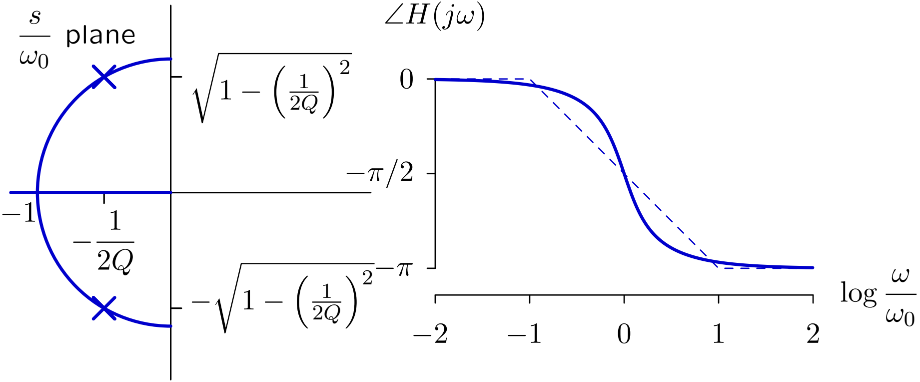


Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$Q = 1$$

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



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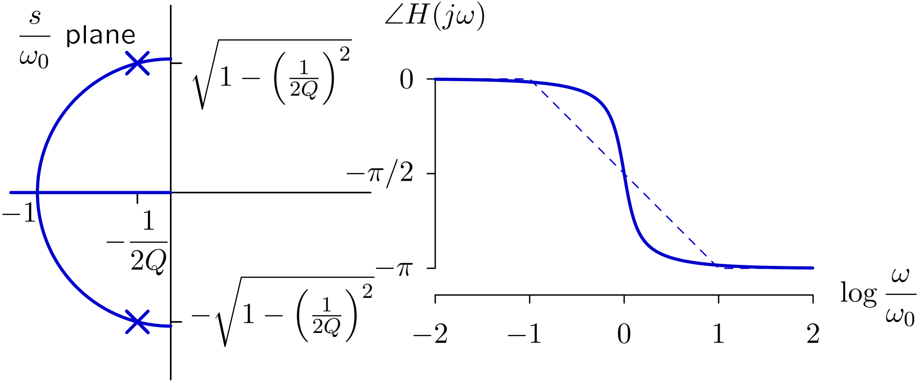

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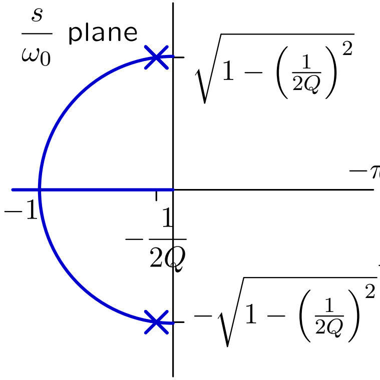

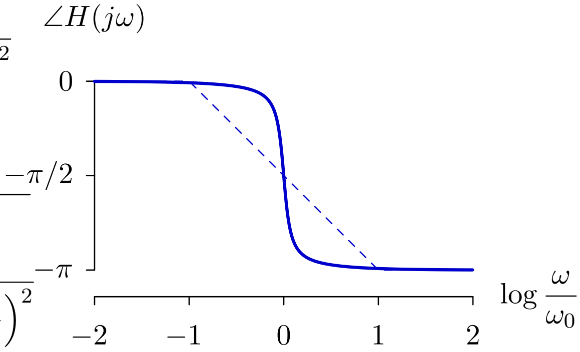

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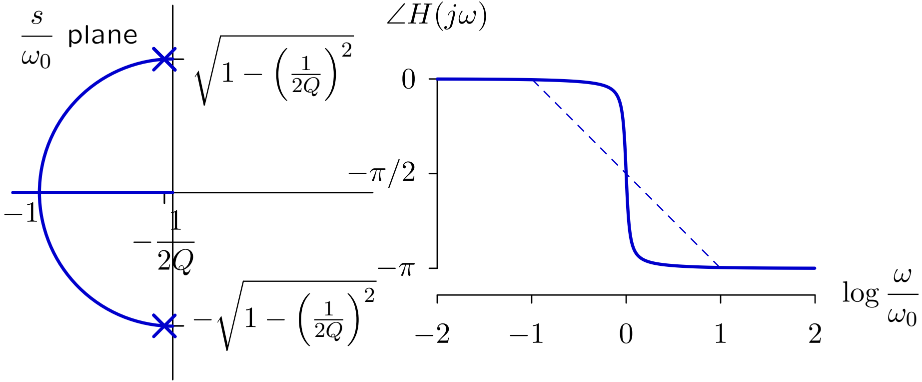
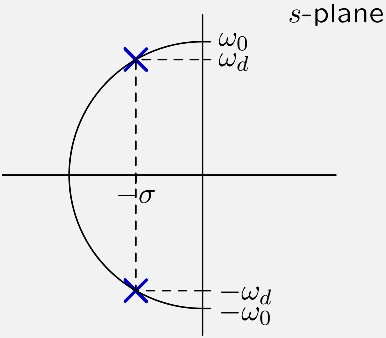


Check Yourself

Consider the system represented by the following poles.

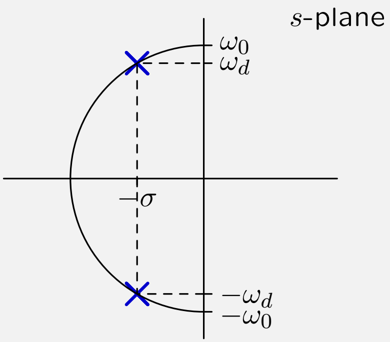


At what frequency ω will the step response oscillate?

1. $\omega > \omega_0$
2. $\omega = \omega_0$
3. $\omega_d < \omega < \omega_0$
4. $\omega = \omega_d$
5. $\omega < \omega_d$

Check Yourself

Consider the system represented by the following poles.

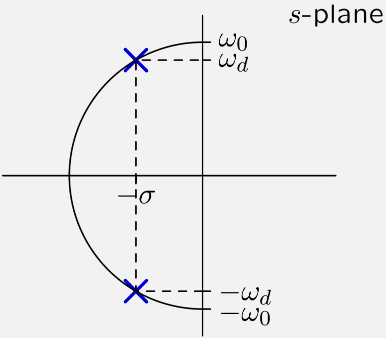


At what frequency ω will the step response oscillate? 4

1. $\omega > \omega_0$
2. $\omega = \omega_0$
3. $\omega_d < \omega < \omega_0$
4. $\omega = \omega_d$
5. $\omega < \omega_d$

Check Yourself

Consider the system represented by the following poles.

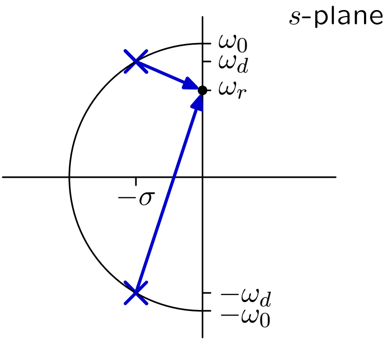


Find the frequency ω_r at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos \omega_r t$.

1. $\omega_r > \omega_0$
2. $\omega_r = \omega_0$
3. $\omega_d < \omega_r < \omega_0$
4. $\omega_r = \omega_d$
5. $\omega_r < \omega_d$

Check Yourself: Frequency Response

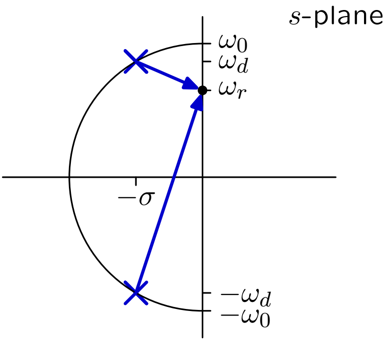
Analyze with vectors.



The product of the lengths is $\left(\sqrt{(\omega_r+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega_d-\omega_r)^2+\sigma^2}\right)$.

Check Yourself: Frequency Response

Analyze with vectors.

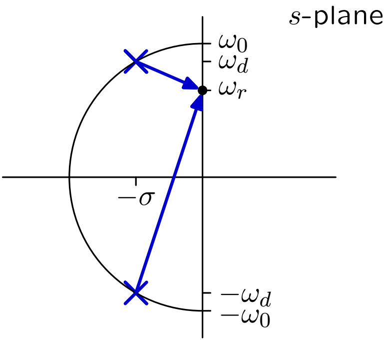


The product of the lengths is $\left(\sqrt{(\omega_r+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega_d-\omega_r)^2+\sigma^2}\right)$.

Decreasing ω_r from ω_d to $\omega_d-\epsilon$ decreases the product since length of bottom vector decreases as ϵ while length of top vector increases only as ϵ^2.

Check Yourself: Frequency Response

More mathematically ...



The product of the lengths is $\left(\sqrt{(\omega_r+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega_r-\omega_d)^2+\sigma^2}\right)$.

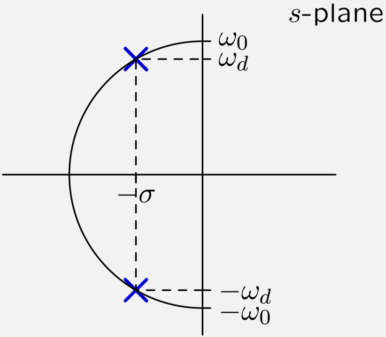
Maximum occurs where derivative of squared lengths is zero.

$$\frac{d}{d\omega_r}\left((\omega_r+\omega_d)^2+\sigma^2\right)\left((\omega_r-\omega_d)^2+\sigma^2\right)=0$$

$$\rightarrow \quad \omega_r^2=\omega_d^2-\sigma^2=\omega_0^2-2\sigma^2\,.$$

Check Yourself

Consider the system represented by the following poles.

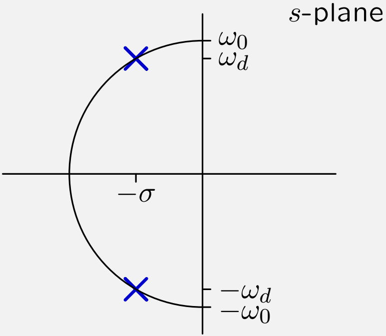


Find the frequency ω_r at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos\omega_r t$. 5

1. $\omega_r > \omega_0$
2. $\omega_r = \omega_0$
3. $\omega_d < \omega_r < \omega_0$
4. $\omega_r = \omega_d$
5. $\omega_r < \omega_d$

Check Yourself

The following poles characterize a resonant system.



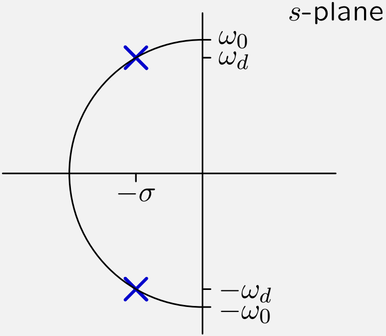
What is the 3 dB bandwidth of the resonant peak?

1. $\sigma/2$
2. σ
3. $\omega_0 - \omega_d$
4. $2(\omega_0 - \omega_d)$

None of the above

Check Yourself

The following poles characterize a resonant system.



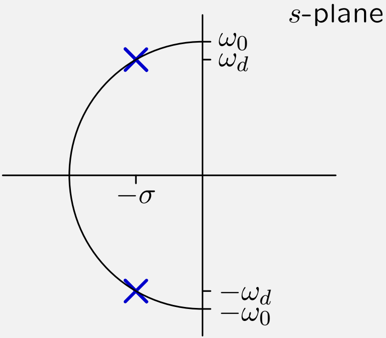
What is the 3 dB bandwidth of the resonant peak? 2

1. $\sigma/2$
2. σ
3. $\omega_0 - \omega_d$
4. $2(\omega_0 - \omega_d)$

None of the above

Check Yourself

Consider the system represented by the following poles.

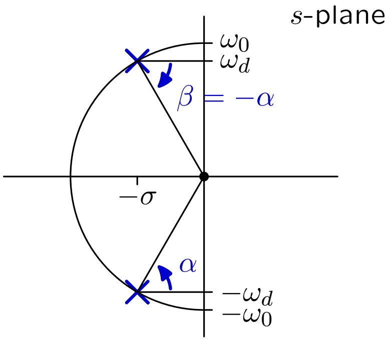


Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos\omega t$.

1. $\omega > \omega_0$
2. $\omega = \omega_0$
3. $\omega_d < \omega < \omega_0$
4. $\omega = \omega_d$
5. $\omega < \omega_d$

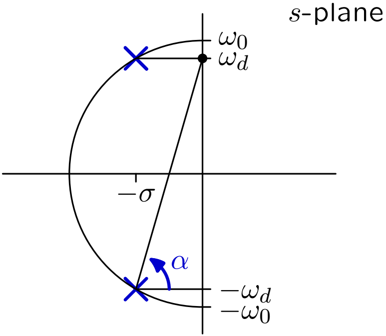
Check Yourself

The phase is 0 when $\omega = 0$.



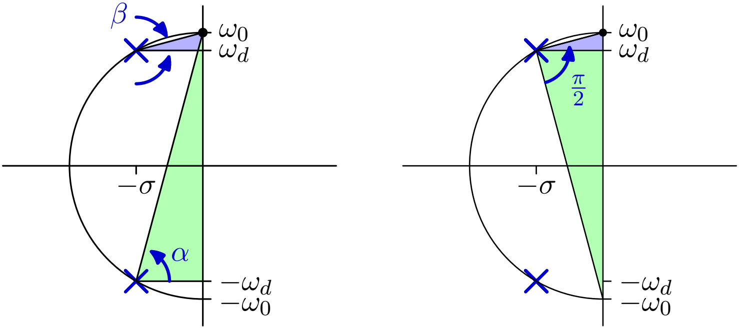
Check Yourself

The phase hasn't reached $-\pi/2$ when $\omega = \omega_d$.



Check Yourself

The phase is $-\pi/2$ at $\omega = \omega_0$.



Check Yourself

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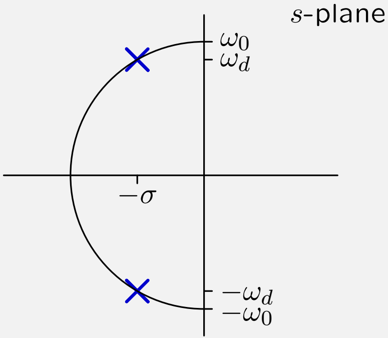

Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos\omega t$. 2

1. $\omega > \omega_0$
2. $\omega = \omega_0$
3. $\omega_d < \omega < \omega_0$
4. $\omega = \omega_d$
5. $\omega < \omega_d$

Summary

The frequency response of a system is easily determined using Bode plots.

Each pole and each zero contributes one section to the Bode plot.

The magnitude of the response of the system is given by the sum of the log magnitudes for the sections contributed by each pole and zero.

The angle of the response of the system is given by the sum of the angles for the sections contributed by each pole and zero.