Gain Margins, Phase Margins, and Lead Compensation
Controller Design

Goal: Given a system $H(s)$ (the plant), design a controller $K(s)$ to achieve some set of performance goals.

$$\begin{align*}
X & \rightarrow + \rightarrow K(s) \rightarrow H(s) \rightarrow Y = G(s)X
\end{align*}$$

The goals may be specified in the time domain and/or frequency domain.

The time domain objectives include:
- overshoot
- steady-state
- convergence

The frequency domain objectives include:
- $|G(j\omega)|$
- $Q$
- $\omega_r$
To date, we have focused on PID controllers. All of our controllers included a proportional term. Adding a derivative term can increase stability. Adding an integral term can decrease steady-state errors. Derivative and integral are time-domain descriptions. Today: focus on frequency-domain representations of controllers.
Check Yourself

Consider the magnitude of the frequency responses of four possible controllers.

Which could correspond to a PID controller?

\[ K(s) = K_p + sK_d + \frac{K_i}{s} \]
Consider the magnitude of the frequency responses of four possible controllers.

\[
K(s) = K_p + sK_d + \frac{K_i}{s}
\]

If \(K_d\) is nonzero, then the frequency response is large at high frequencies.
If \(K_i\) is nonzero, then the frequency response is large at low frequencies.
\(\rightarrow K_2(s)\)
Check Yourself

Consider the magnitude of the frequency responses of four possible controllers.

\[
|K_1(s)|_{[\text{dB}]} \\
0 \quad 0 \quad 1 \quad 10 \quad 100 \quad \omega \text{ [log]}
\]

\[
|K_2(s)|_{[\text{dB}]} \\
0 \quad 0 \quad 1 \quad 10 \quad 100 \quad \omega \text{ [log]}
\]

\[
|K_3(s)|_{[\text{dB}]} \\
0 \quad 0 \quad 1 \quad 10 \quad 100 \quad \omega \text{ [log]}
\]

\[
|K_4(s)|_{[\text{dB}]} \\
0 \quad 0 \quad 1 \quad 10 \quad 100 \quad \omega \text{ [log]}
\]

Which could correspond to a PID controller? 2

\[
K(s) = K_p + sK_d + \frac{K_i}{s}
\]

Are there other useful types of controllers?
Stability Criteria

To be useful, a controller must make the closed-loop system stable.

\[ X \rightarrow + \rightarrow K_p \rightarrow H(s) \rightarrow Y \]

Under what conditions will the closed-loop system be stable?
Check Yourself

To be useful, a controller must make the closed-loop system stable.

\[ G(s) = \frac{Y}{X} \]

Which (if any) of the following statements are true?

- If \( H(s) \) has a pole in the right-half plane, then \( G(s) \) is unstable.
- If \( H(s) \) has just two poles (\( s = \pm j\omega_0 \)), \( G(s) \) will be stable if \( K_p > 1 \).
- If \( K_p H(s) = -1 \) for \( s = j\omega_0 \), then the system cannot be stable.
Check Yourself

To be useful, a controller must make the closed-loop system stable.

Can the closed-loop system be stable if $H(s)$ has a pole in the right-half plane?

Try a simple example: $H(s)$ has a single pole at $s = 1$.

$$H(s) = \frac{1}{s - 1}$$

$$G(s) = \frac{Y}{X} = \frac{\frac{K_p}{s-1}}{1 + \frac{K_p}{s-1}} = \frac{K_p}{s - 1 + K_p}$$

The closed-loop pole $s = 1 - K_p$ will be in the left half plane if $K_p > 1$.

→ The closed-loop system can be stable even if the open-loop system is unstable.
Check Yourself

To be useful, a controller must make the closed-loop system stable.

\[ X \rightarrow \begin{array}{c} + \end{array} \rightarrow \begin{array}{c} K_p \end{array} \rightarrow \begin{array}{c} H(s) \end{array} \rightarrow Y \]

\[ G(s) = \frac{Y}{X} \]

Which (if any) of the following statements are true?

- If \( H(s) \) has a pole in the right-half plane, then \( G(s) \) is unstable.  \( \times \)
- If \( H(s) \) has just two poles \((s=\pm j\omega_0)\), \( G(s) \) will be stable if \( K_p > 1 \).
- If \( K_pH(s) = -1 \) for \( s = j\omega_0 \), then the system cannot be stable.
Check Yourself

To be useful, a controller must make the closed-loop system stable.

Can the closed-loop system be stable if $H(s)$ has poles at $s = \pm j\omega_0$?

\[
H(s) = \frac{1}{(s - j\omega_0)(s + j\omega_0)} = \frac{1}{s^2 + \omega_0^2}
\]

\[
G(s) = \frac{Y}{X} = \frac{\frac{K_p}{s^2 + \omega_0^2}}{1 + \frac{K_p}{s^2 + \omega_0^2}} = \frac{K_p}{s + \omega_0^2 + K_p}
\]

If $K_p > 1$ the closed-loop poles are on the $j\omega$ axis.

$\rightarrow$ The system is unstable for $K_p > 1$. Feedback did not stabilize this system.
Check Yourself

To be useful, a controller must make the closed-loop system stable.

\[ G(s) = \frac{Y}{X} \]

Which (if any) of the following statements are true?

- If \( H(s) \) has a pole in the right-half plane, then \( G(s) \) is unstable.  \( \times \)
- If \( H(s) \) has just two poles \( (s=\pm j\omega_0) \), \( G(s) \) will be stable if \( K_p > 1 \).  \( \times \)
- If \( K_p H(s) = -1 \) for \( s = j\omega_0 \), then the system cannot be stable.
Check Yourself

To be useful, a controller must make the closed-loop system stable.

Can the system be stable if \( K_pH(j\omega_0) = -1 \)?

\[
G(j\omega_0) = \frac{K_pH(j\omega_0)}{1 + K_pH(j\omega_0)} = \frac{-1}{1 - 1} \to \infty
\]

The system has a pole on the \( j\omega \) axis. The system cannot be stable.
Check Yourself

To be useful, a controller must make the closed-loop system stable.

\[ G(s) = \frac{Y}{X} \]

Which (if any) of the following statements are true?

- If \( H(s) \) has a pole in the right-half plane, then \( G(s) \) is unstable.  \( \times \)
- If \( H(s) \) has just two poles \( (s = \pm j\omega_0) \), \( G(s) \) will be stable if \( K_p > 1 \).  \( \times \)
- If \( K_p H(s) = -1 \) for \( s = j\omega_0 \), then the system cannot be stable.  \( \sqrt \)

This last condition is the basis of lead compensation (today) and root locus methods (next time).
Determining Stability from Open-Loop Frequency Response

There is a closed-loop pole at every frequency $\omega_0$ for which $K_p H(j\omega_0) = -1$.

From Black’s equation,

$$ G(j\omega_0) = \frac{K_p H(j\omega_0)}{1 + K_p H(j\omega_0)} $$

If $K_p H(j\omega_0) = -1$, then $|G(j\omega_0)| \to \infty$

But $G(s)$ can also be written as a ratio of first-order factors:

$$ G(s) = K \frac{(s - z_1)(s - z_2)(s - z_3) \cdots}{(s - p_1)(s - p_2)(s - p_3) \cdots} $$

and if $G(s) \to \infty$ then $j\omega_0$ is a root of the denominator.

The closed-loop system $G(s)$ must have a pole at $s = j\omega_0$. 
Determining Stability from Open-Loop Frequency Response

Consider the frequency response of an open-loop system $H(s) = \frac{1}{s(s+1)(s+5)}$.

Is there a frequency $\omega$ at which $H(j\omega) = -1$?
Determining Stability from Open-Loop Frequency Response

Consider the frequency response of an open-loop system $H(s) = \frac{1}{s(s+1)(s+5)}$.

$H(j\omega) = -1$? No. $|H(j\omega_1)| = 1$ and $\angle(H(j\omega_2)) = -\pi$ but $\omega_1 \neq \omega_2$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0)) = -\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 1$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 2$

![Graph showing the magnitude and phase of $H(j\omega)$, and a closed-loop step response.]
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise. $K_p = 5$

- $K_p$ versus $\omega$ [log scale]
- $|H(j\omega)|$ [dB] versus $\omega$ [log scale]
- $\angle(H(j\omega))$ [rad.], $\omega$ [log scale]

Closed-loop step response:

- $t$ [s] from 0 to 15
Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise. 

$K_p = 10$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 20$

![Graph showing magnitude and phase of $H(j\omega)$ against frequency on a log scale.](image)

![Graph showing closed-loop step response against time.](image)
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise. $K_p = 30$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0)) = -\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 32$

![Diagram of frequency response and step response](image-url)
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 33$

<table>
<thead>
<tr>
<th>$K_p$</th>
<th>$H(j\omega)$</th>
<th>dB</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>40</td>
<td>-40</td>
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<td>-40</td>
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<tr>
<td>-60</td>
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<table>
<thead>
<tr>
<th>$\omega$ [log scale]</th>
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<tr>
<td>0.01 0.1 1 10 100</td>
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<tr>
<th>$\angle(H(j\omega))$ [rad.]</th>
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<tr>
<td>0 $-\pi/2$ $-\pi$ $-3\pi/2$</td>
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<th>$t$ [s]</th>
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<td>0 5 10 15</td>
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closed-loop step response
Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise. $K_p = 1$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 2$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0)) = -\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise. $K_p = 5$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 10$

![Graph showing Bode plots for frequency response and closed-loop step response]
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise. $K_p = 20$

![Gain margin and phase margin diagrams](image_url)

Closed-loop step response.
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0)) = -\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 30$

![Graph showing Bode plots and step response](image)
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $\angle(H(j\omega_0)) = -\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise. $K_p = 32$

![Graph showing gain margin and phase response]
Determining Stability from Open-Loop Frequency Response

Let \( \omega_0 \) represent the frequency where \( \angle(H(j\omega_0)) = -\pi \). The system will be stable if the magnitude of \( H(j\omega_0) \) is less than 1 and unstable otherwise.

\[ K_p = 33 \]

\[ \begin{align*}
\text{Gain Margin} \quad \omega \text{ [log scale]} \\
\text{Closed-loop Step Response} \quad t \text{ [s]}
\end{align*} \]
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise. $K_p = 1$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise. $K_p = 2$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise. $K_p = 5$
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$K_p = 10$
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise. $K_p = 20$

\[\begin{align*}
|H(j\omega)| &\quad \text{[dB]} \\
K_p &\quad \omega \quad \text{[log scale]} \\
\angle(H(j\omega)) &\quad \text{[rad.]} \\
\omega \quad \text{[log scale]} &\quad 0.01 \\
&\quad 0.1 \\
&\quad 1 \\
&\quad 10 \\
&\quad 100 \\
\end{align*}\]
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$k_p = 30$

[Graphs showing $K_p |H(j\omega)|$ in dB, $\angle H(j\omega)$ in rad, and closed-loop step response.]
Determining Stability from Open-Loop Frequency Response

Let \( \omega_0 \) represent the frequency where \( |H(j\omega_0)| = 1 \). The system will be stable if the angle of \( H(j\omega_0) \) is greater than \(-\pi\) and unstable otherwise.

\[ K_p = 32 \]
Determining Stability from Open-Loop Frequency Response

Let $\omega_0$ represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 33$

![Graph showing Bode plot and Nyquist plot with phase margin indicated.](image-url)
Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

\[ K_p = 1 \]

- **Gain margin**
  \[ \left| K_p \right| = \frac{1}{\omega} \text{[dB]} \]
  \[ ω \text{[log scale]} \]

- **Phase margin**
  \[ \angle \left( \frac{1}{H(jω)} \right) \] [rad.]
  \[ ω \text{[log scale]} \]

Closed-loop step response
Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

\[ K_p = 2 \]

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**Gain Margin**

- \( K_p \) vs \( \log{\omega} \)
- \( |H(j\omega)| \) in [dB]

**Phase Margin**

- \( \angle(H(j\omega)) \) in [rad.]
- \( \log{\omega} \)

**Closed-Loop Step Response**

- \( t \) in [s]
- \( t \) range: 0 to 15

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**Graphical Illustration**

- Gain margin:
  - Frequency range: \( 0.01 \) to \( 100 \)
  - DB scale: \(-40\) to \(-60\)
- Phase margin:
  - Frequency range: \( 0.01 \) to \( 100 \)
  - Radian scale: \(-3\pi/2\) to \(-\pi\)
Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

\[ K_p = 5 \]
Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

\[ K_p = 10 \]

![Graph showing gain and phase margins with open-loop frequency response and closed-loop step response.](image)
Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response. 

\( K_p = 20 \)

[Graph of open-loop frequency response showing gain margin and phase margin]

[Graph of closed-loop step response]
Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

\[ K_p = 30 \]
Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

\[ K_p = 32 \]

![Gain and phase margin diagram](image)

- **Gain Margin**: \[ |K_p| H(j\omega) \] dB
- **Phase Margin**: \( \angle \left( H(j\omega) \right) \) rad.

![Closed-loop step response](image)

- **Closed-loop Step Response**: Time response of the system.
Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

\[ K_p = 33 \]

![Gain Margin](image1)

![Phase Margin](image2)

![Closed-Loop Step Response](image3)
Lead Compensation

Stability can be enhanced by increasing the gain and/or phase margin using a compensator as shown below.

We can use a lead compensator to increase the phase margin.

\[ L(s) = \left( \frac{p}{z} \right) \left( \frac{s + z}{s + p} \right) \]
A lead compensator has no effect on the magnitude or phase at low frequencies.

\[ L(s) = \left( \frac{p}{z} \right) \left( \frac{s + z}{s + p} \right) \]
A lead compensator can significantly increase phase margin (which is good). Unfortunately, it also reduces the gain margin a bit (which is not so good).

\[ L(s) = \left( \frac{p}{z} \right) \left( \frac{s+z}{s+p} \right) \]

When adjusted appropriately, the increase in phase margin can more than compensate for the slight loss of gain margin.
Improving Performance with Lead Compensation

Using a lead compensator with $z = 20$ and $p = 200$ has a very small effect. $K_p = 20 
\omega$ [log scale] $\angle [\text{rad.}] 
\omega$ [log scale] 
closed-loop step response
Improving Performance with Lead Compensation

Moving the compensator to a lower frequency increases convergence rate.

\[ K_p = 20 \]
\[ z = 10; \quad p = 100 \]
Improving Performance with Lead Compensation

Moving the compensator to a lower frequency increases convergence rate.

\[ K_p = 20 \]
\[ z = 5; \quad p = 50 \]
Convergence is dramatically improved when $z = 2$ and $p = 20$.

$K_p = 20$

$z = 2; \quad p = 20$
Improving Performance with Lead Compensation

Convergence for $z = 1$ not as good as $z = 2$ — now losing gain margin.

$K_p = 20$

$z = 1; \ p = 10$

![Graphs showing Bode plots and closed-loop step response for different values of $z$.](image-url)
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.5$.

$K_p = 20$

$z = 0.5; \quad p = 5$

![Graphs showing Bode plots and a closed-loop step response.](image)
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.4$.

$K_p = 20$

$z = 0.4; \quad p = 4$

![Graph showing the Bode plots and closed-loop step response for a system with lead compensation.](image)
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.35$.

$K_p = 20$

$z = 0.35; \quad p = 3.5$

![Graph showing frequency response](image)

![Closed-loop step response](image)
Improving Performance with Lead Compensation

The system is unstable when \( z = 0.34 \).

\[ K_p = 20 \]

\( z = 0.34; \quad p = 3.4 \)
Check Yourself

What is the relation (if any) between lead compensation and PD control?

Lead compensation and PD control are ...
- equivalent if the zero in the lead compensator is at infinity
- equivalent if the pole in the lead compensator is at infinity
- equivalent if the zero in the lead compensator is at \(-\frac{K_p}{K_d}\)
- equivalent if the zero in the lead compensator is at \(-\frac{K_p}{K_d}\) and the pole in the lead compensator is at infinity
- never equivalent
Check Yourself

PID control

![PID Control Graph](image)

lead compensation

![Lead Compensation Graph](image)
Summary

Today we focused on a frequency-response approach to controller design.

Stability criterion: Let $\omega_0$ represent the frequency at which the open-loop phase is $-\pi$. The closed loop system will be stable if the magnitude of the open-loop system at $\omega_0$ is less than 1.

Useful metrics for characterizing relative stability:

- gain margin: ratio of the maximum stable gain to the current gain
- phase margin: additional phase lag needed to make system unstable

Lead compensation can improve performance by increasing phase margin (while also decreasing gain margin slightly).

Next time: root-locus method.