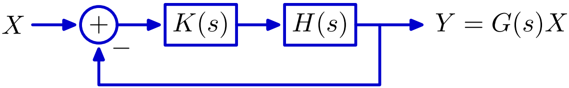
6.3100: Dynamic System Modeling and Control Design

Gain Margins, Phase Margins, and Lead Compensation

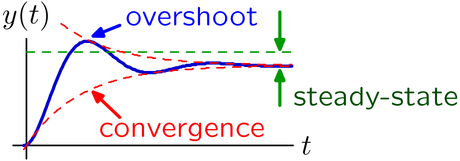
March 18, 2024

Controller Design

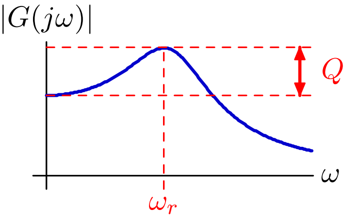
Goal: Given a system $H(s)$ (the plant), design a controller $K(s)$ to achieve some set of performance goals.



The goals may be specified in the time domain



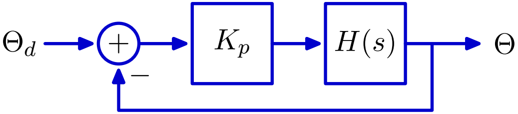
and/or frequency domain.



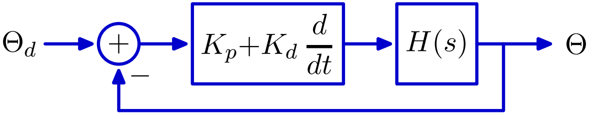
PID Controllers

To date, we have focused on PID controllers.

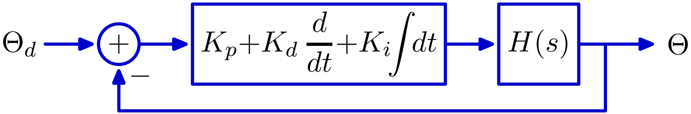
All of our controllers included a proportional term.



Adding a derivative term can increase stability.



Adding an integral term can decrease steady-state errors.

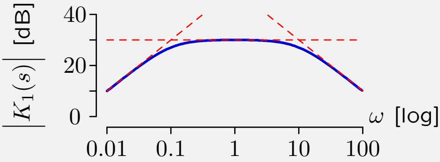


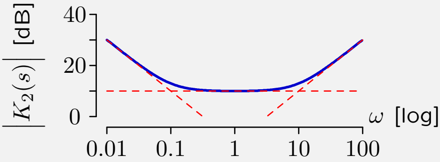
Derivative and integral are **time-domain** descriptions.

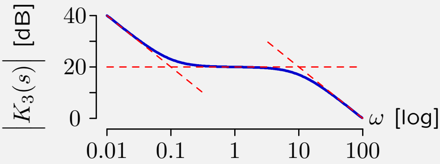
Today: focus on **frequency-domain** representations of controllers.

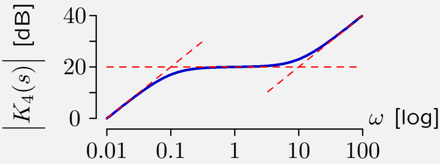
Check Yourself

Consider the magnitude of the frequency responses of four possible controllers.







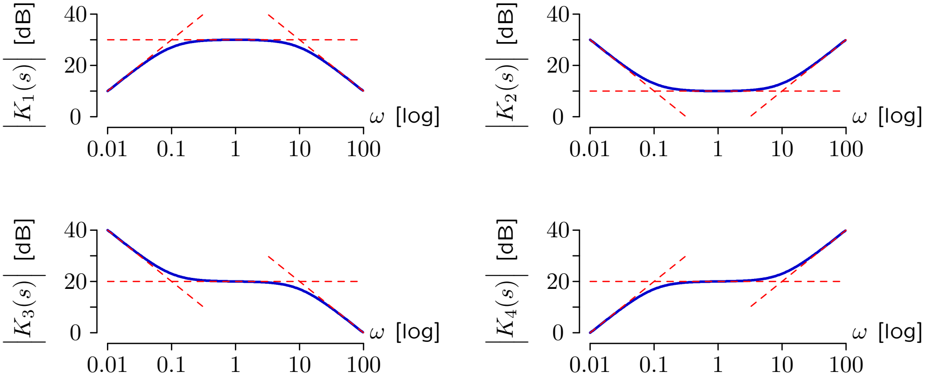


Which could correspond to a PID controller?

$$K(s) = K_p + sK_d + \frac{K_i}{s}$$

Check Yourself

Consider the magnitude of the frequency responses of four possible controllers.

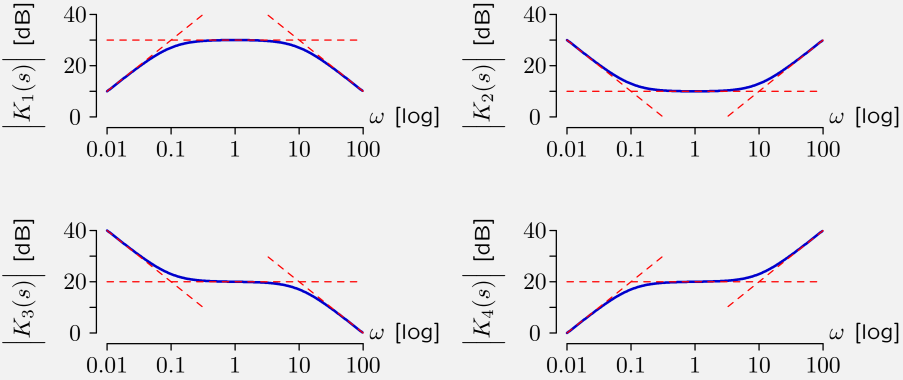


$$K(s) = K_p + sK_d + \frac{K_i}{s}$$

If K_d is nonzero, then the frequency response is large at high frequencies.
If K_i is nonzero, then the frequency response is large at low frequencies.
$\rightarrow K_2(s)$

Check Yourself

Consider the magnitude of the frequency responses of four possible controllers.



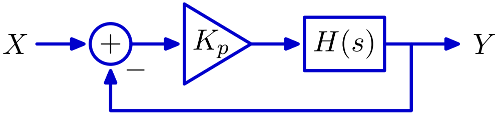
Which could correspond to a PID controller? 2

$$K(s) = K_p + sK_d + \frac{K_i}{s}$$

Are there other useful types of controllers?

Stability Criteria

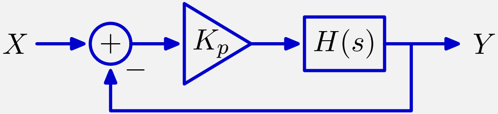
To be useful, a controller must make the closed-loop system stable.



Under what conditions will the closed-loop system be stable?

Check Yourself

To be useful, a controller must make the closed-loop system stable.



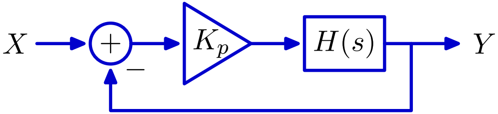
$$G(s) = \frac{Y}{X}$$

Which (if any) of the following statements are true?

- If $H(s)$ has a pole in the right-half plane, then $G(s)$ is unstable.
- If $H(s)$ has just two poles ($s = \pm j\omega_0$), $G(s)$ will be stable if $K_p > 1$.
- If $K_p H(s) = -1$ for $s = j\omega_0$, then the system cannot be stable.

Check Yourself

To be useful, a controller must make the closed-loop system stable.



Can the closed-loop system be stable if $H(s)$ has a pole in the right-half plane?

Try a simple example: $H(s)$ has a single pole at $s = 1$.

$$H(s) = \frac{1}{s-1}$$

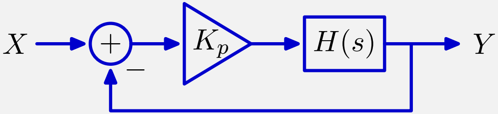
$$G(s) = \frac{Y}{X} = \frac{\frac{Kp}{s-1}}{1+\frac{Kp}{s-1}} = \frac{K_p}{s-1+K_p}$$

The closed-loop pole $s = 1 - K_p$ will be in the left half plane if $K_p > 1$.

$\rightarrow$ The closed-loop system can be stable even if the open-loop system is unstable.

Check Yourself

To be useful, a controller must make the closed-loop system stable.



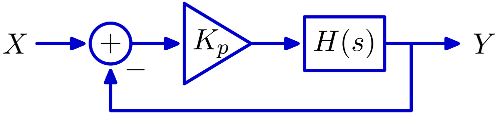
$$G(s) = \frac{Y}{X}$$

Which (if any) of the following statements are true?

- If $H(s)$ has a pole in the right-half plane, then $G(s)$ is unstable. ✗
- If $H(s)$ has just two poles ($s = \pm j\omega_0$), $G(s)$ will be stable if $K_p > 1$.
- If $K_p H(s) = -1$ for $s = j\omega_0$, then the system cannot be stable.

Check Yourself

To be useful, a controller must make the closed-loop system stable.



Can the closed-loop system be stable if $H(s)$ has poles at $s = \pm j\omega_0$?

$$H(s) = \frac{1}{(s - j\omega_0)(s + j\omega_0)} = \frac{1}{s^2 + \omega_0^2}$$

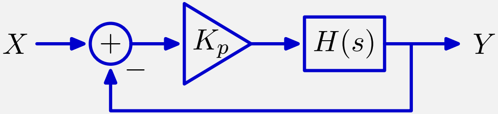
$$G(s) = \frac{Y}{X} = \frac{\frac{K_p}{s^2+\omega_0^2}}{1 + \frac{K_p}{s^2+\omega_0^2}} = \frac{K_p}{s + \omega_0^2 + K_p}$$

If $K_p{>}1$ the closed-loop poles are on the $j\omega$ axis.

$\rightarrow$ The system is unstable for $K_p{>}1$. Feedback did not stabilize this system.

Check Yourself

To be useful, a controller must make the closed-loop system stable.



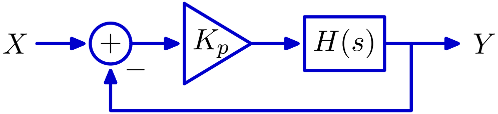
$$G(s) = \frac{Y}{X}$$

Which (if any) of the following statements are true?

- If $H(s)$ has a pole in the right-half plane, then $G(s)$ is unstable. ✗
- If $H(s)$ has just two poles ($s{=}\pm j\omega_0$), $G(s)$ will be stable if $K_p{>}1$. ✗
- If $K_pH(s) = -1$ for $s = j\omega_0$, then the system cannot be stable.

Check Yourself

To be useful, a controller must make the closed-loop system stable.



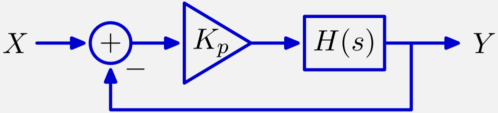
Can the system be stable if $K_pH(j\omega_0) = -1$?

$$G(j\omega_0) = \frac{K_pH(j\omega_0)}{1+K_pH(j\omega_0)} = \frac{-1}{1-1} \to \infty$$

The system has a pole on the $j\omega$ axis. The system cannot be stable.

Check Yourself

To be useful, a controller must make the closed-loop system stable.



$$G(s) = \frac{Y}{X}$$

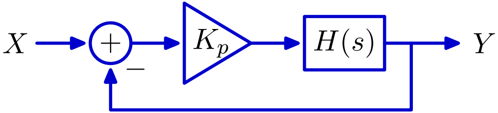
Which (if any) of the following statements are true?

- If $H(s)$ has a pole in the right-half plane, then $G(s)$ is unstable. ✗
- If $H(s)$ has just two poles ($s{=}\pm j\omega_0$), $G(s)$ will be stable if $K_p{>}1$. ✗
- If $K_pH(s) = -1$ for $s = j\omega_0$, then the system cannot be stable. $\surd$

This last condition is the basis of lead compensation (today) and root locus methods (next time).

Determining Stability from Open-Loop Frequency Response

There is a closed-loop pole at every frequency ω_0 for which $K_pH(j\omega_0) = -1$.



From Black's equation,

$$G(j\omega_0) = \frac{K_pH(j\omega_0)}{1 + K_pH(j\omega_0)}$$

If $K_pH(j\omega_0) = -1$, then $|G(j\omega_0)| \to \infty$

But $G(s)$ can also be written as a ratio of first-order factors:

$$G(s) = K\frac{(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

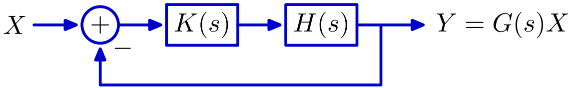
and if $G(s) \to \infty$ then $j\omega_0$ is a root of the denominator.

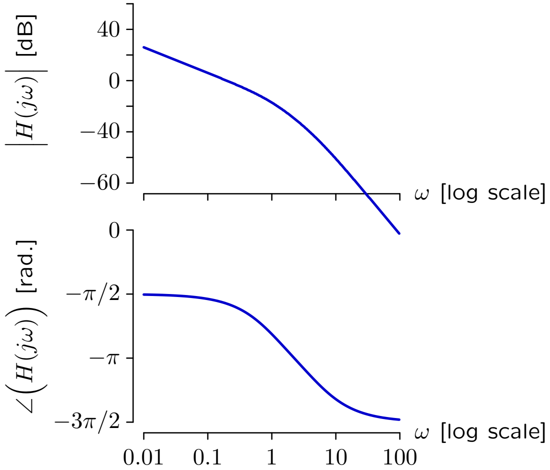
The closed-loop system $G(s)$ must have a pole at $s = j\omega_0$.

Determining Stability from Open-Loop Frequency Response

Consider the frequency response of an open-loop system $H(s) = \frac{1}{s(s+1)(s+5)}$.

Is there a frequency ω at which $H(j\omega) = -1$?

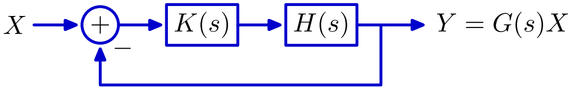


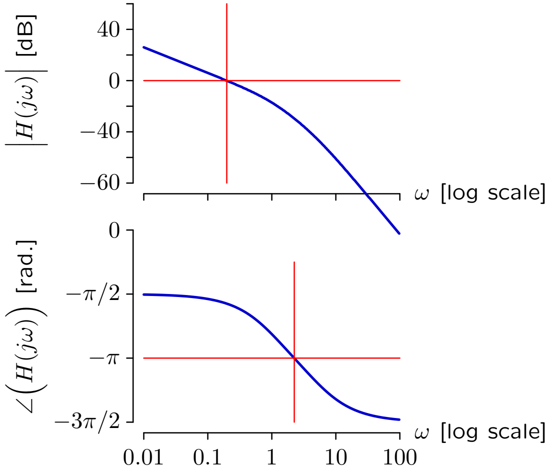


Determining Stability from Open-Loop Frequency Response

Consider the frequency response of an open-loop system $H(s) = \frac{1}{s(s+1)(s+5)}$.

$H(j\omega) = -1$?   No. $|H(j\omega_1)| = 1$ and $\angle(H(j\omega_2)) = -\pi$ but $\omega_1 \neq \omega_2$

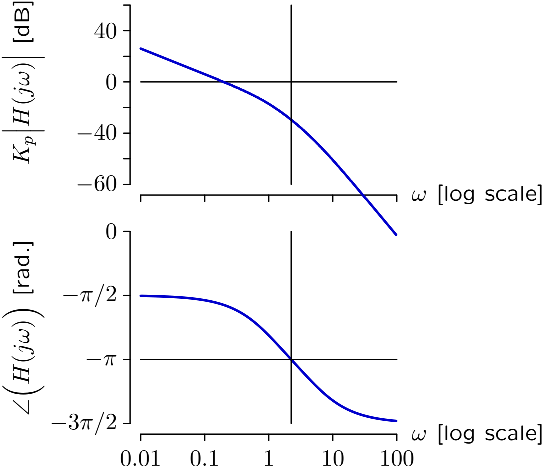


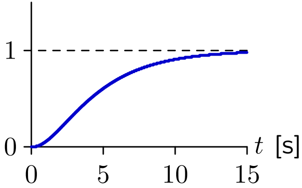


Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 1$



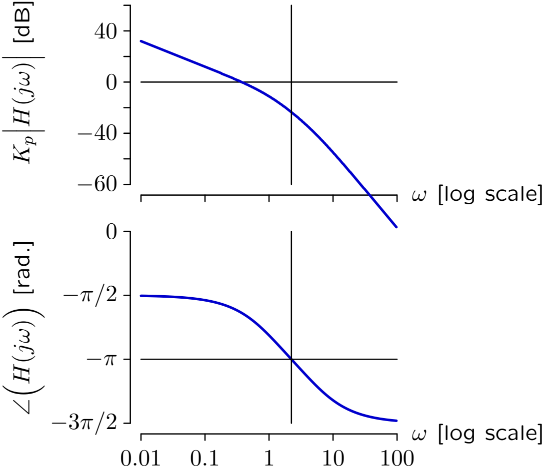


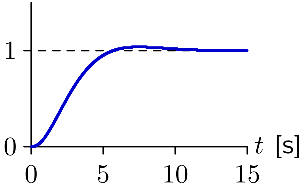
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 2$



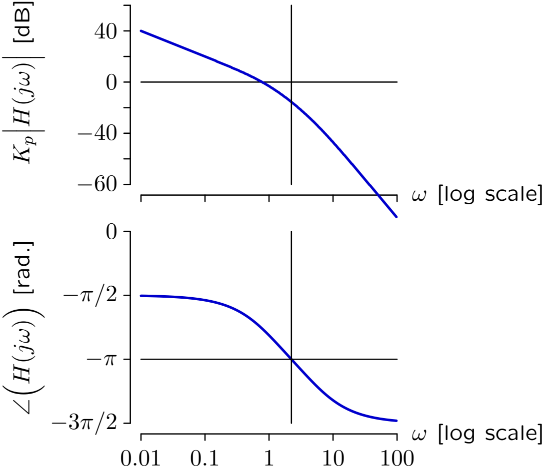


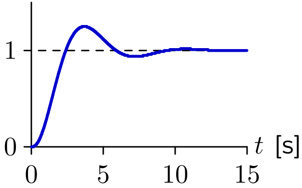
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 5$



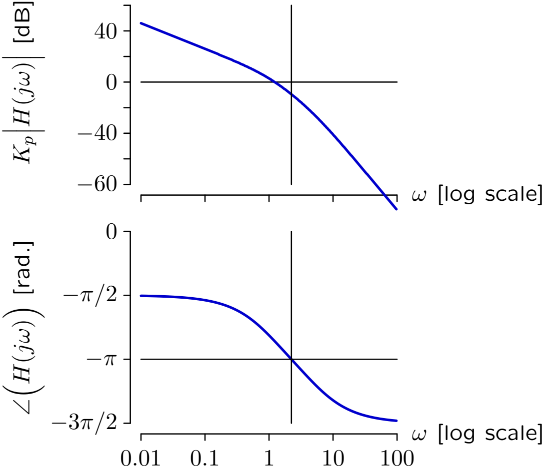


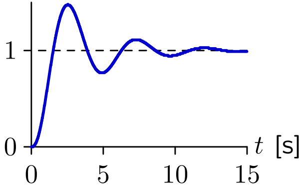
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 10$



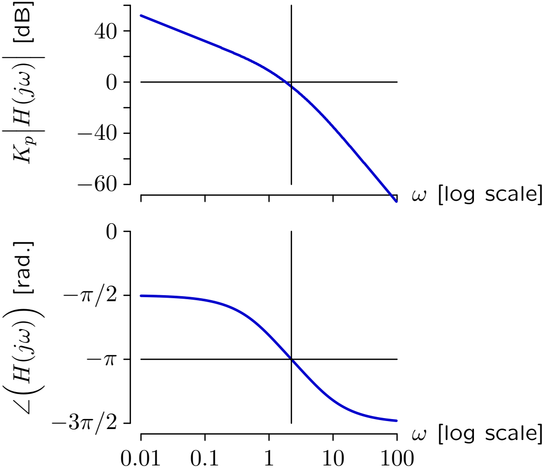


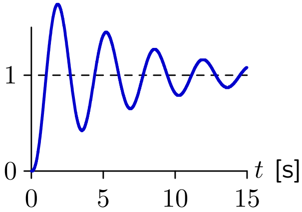
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 20$



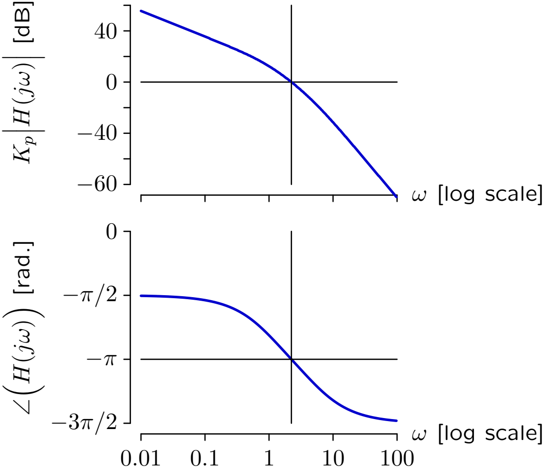


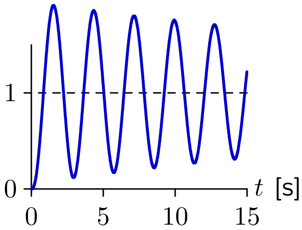
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 30$



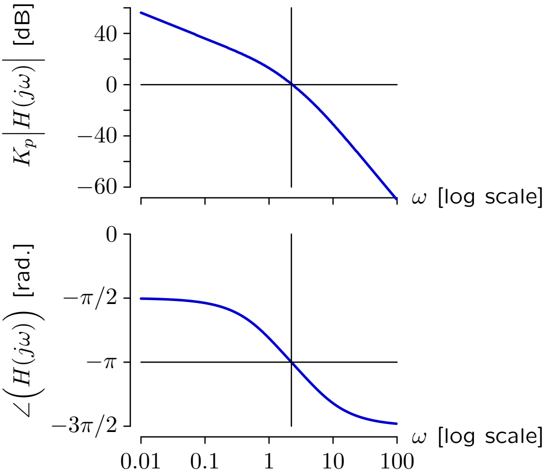


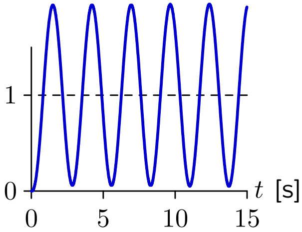
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 32$



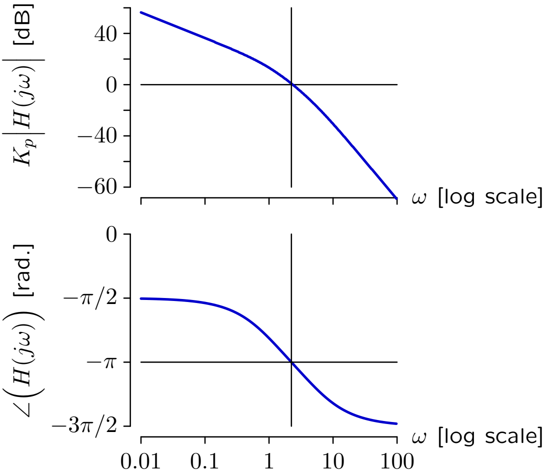


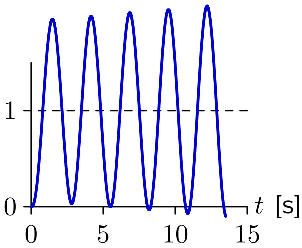
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 33$



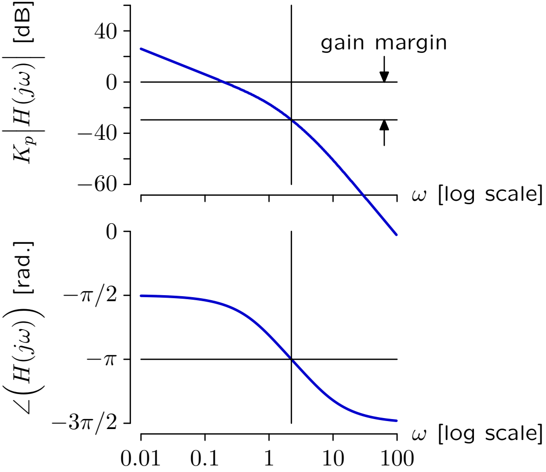


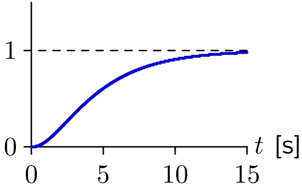
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 1$



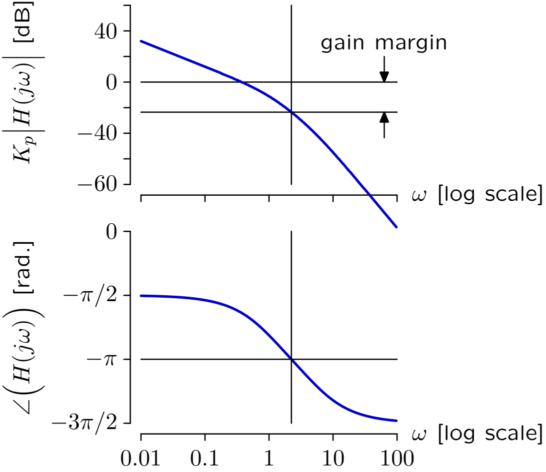


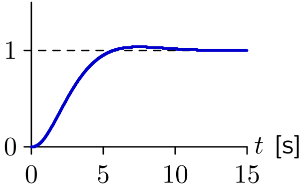
closed-loop step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 2$





closed-loop
step response

Determining Stability from Open-Loop Frequency Response

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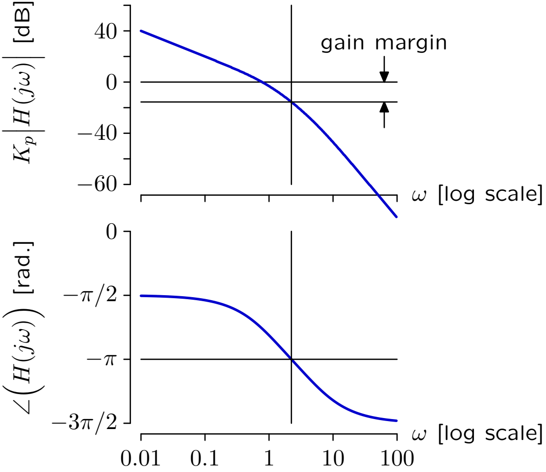
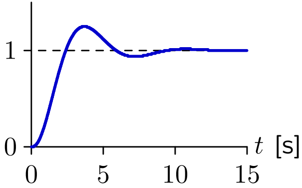
$K_p = 5$





closed-loop
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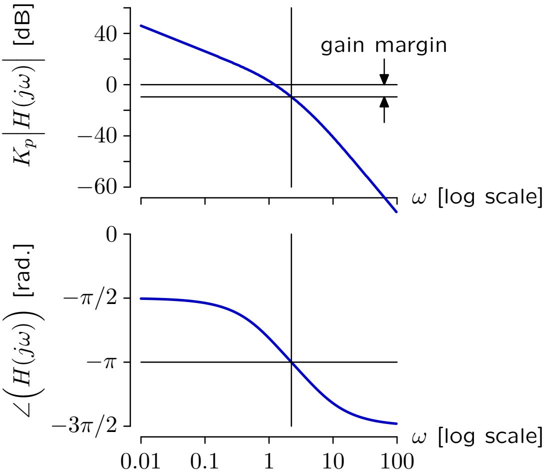
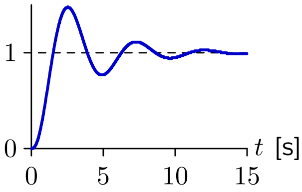


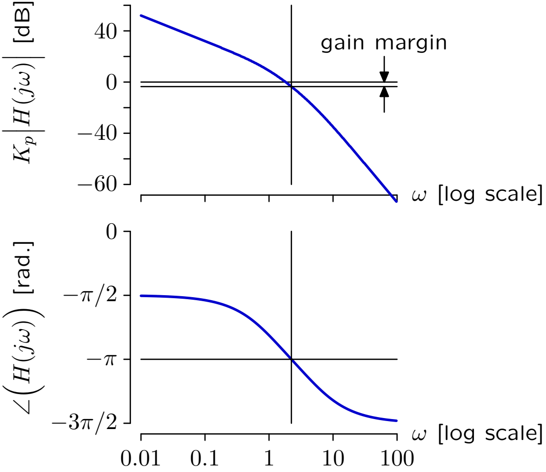


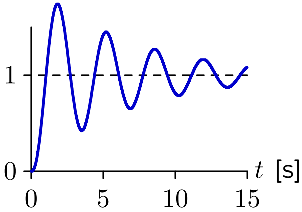
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0))$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 20$



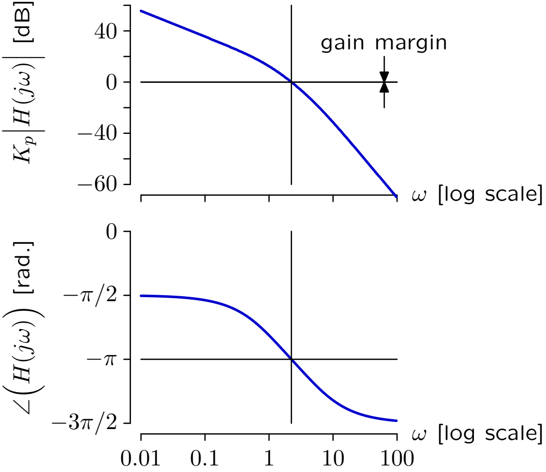


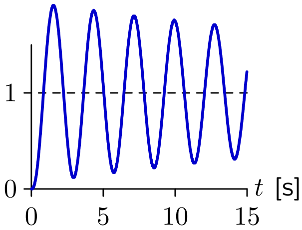
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 30$



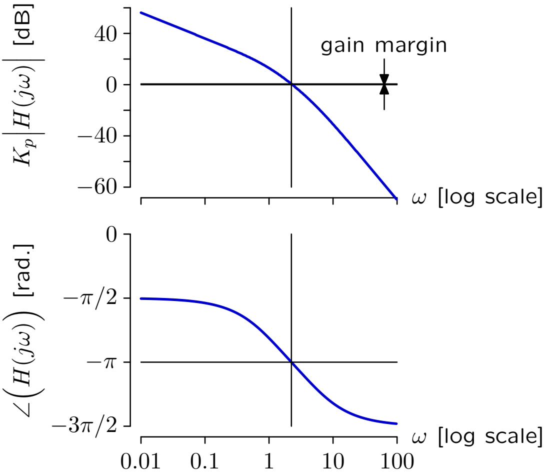


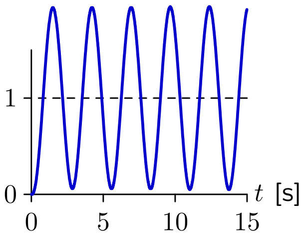
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 32$



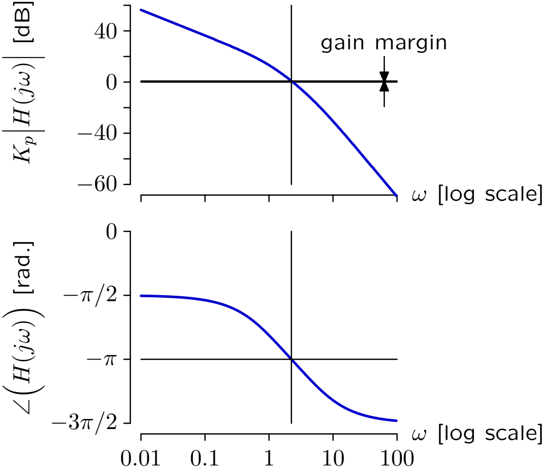


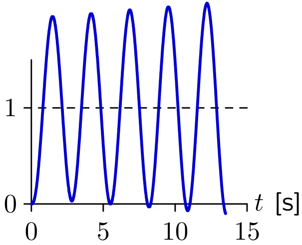
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $\angle(H(j\omega_0)$ is $-\pi$. The system will be stable if the magnitude of $H(j\omega_0)$ is less than 1 and unstable otherwise.

$K_p = 33$



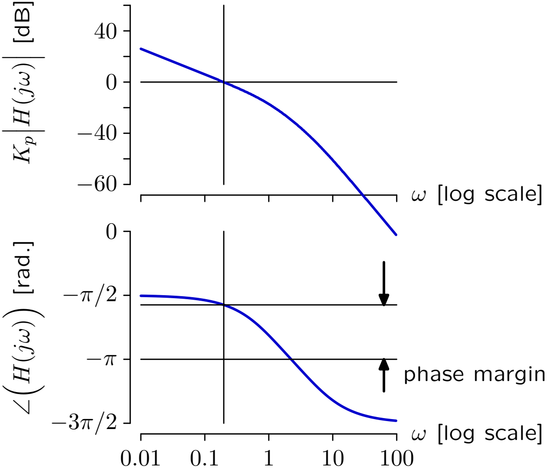


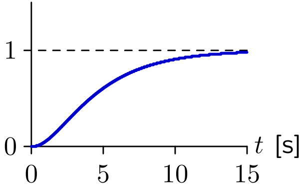
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 1$



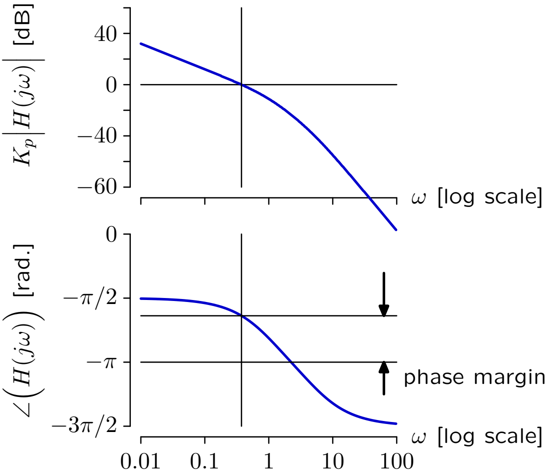


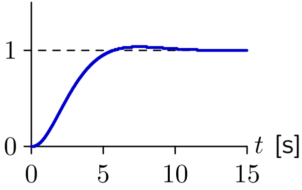
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 2$

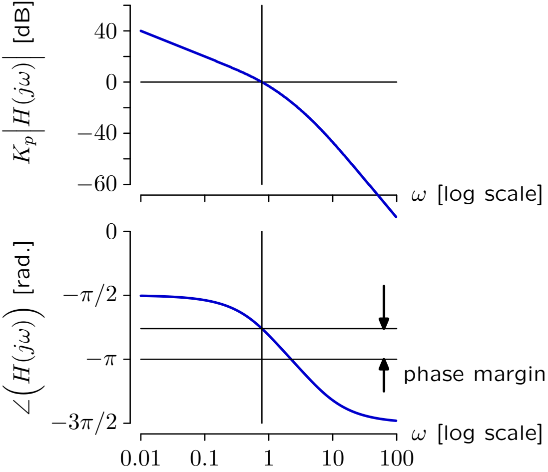


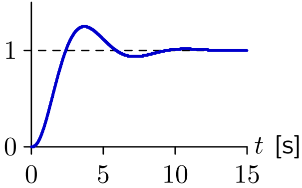


Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 5$



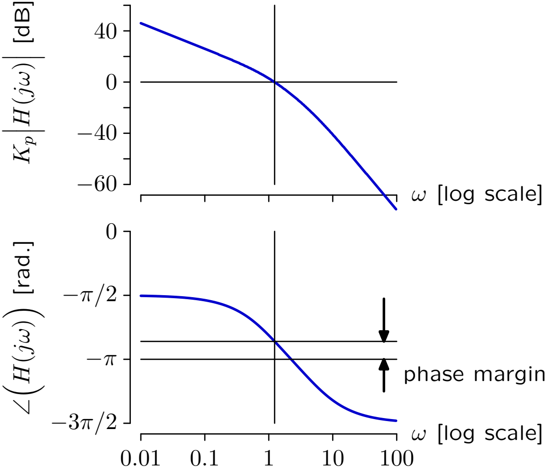


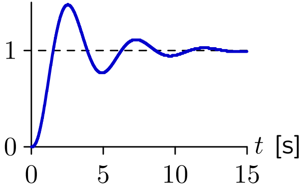
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 10$



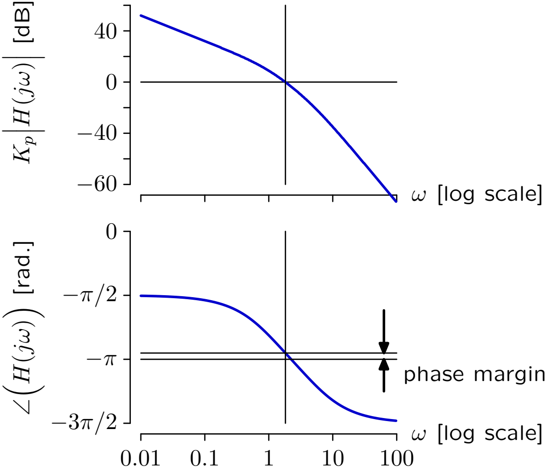


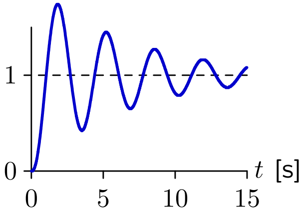
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 20$



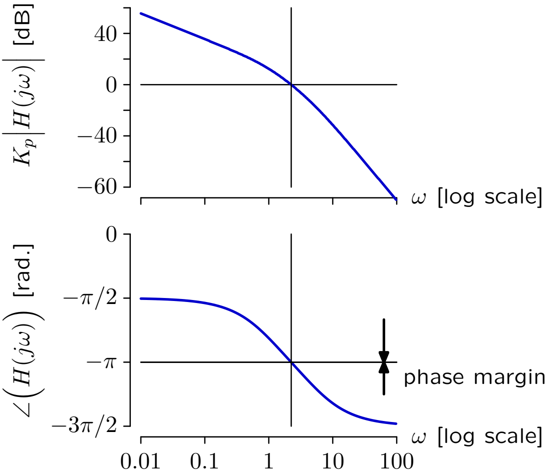


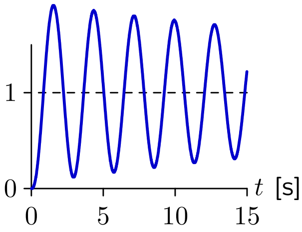
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 30$



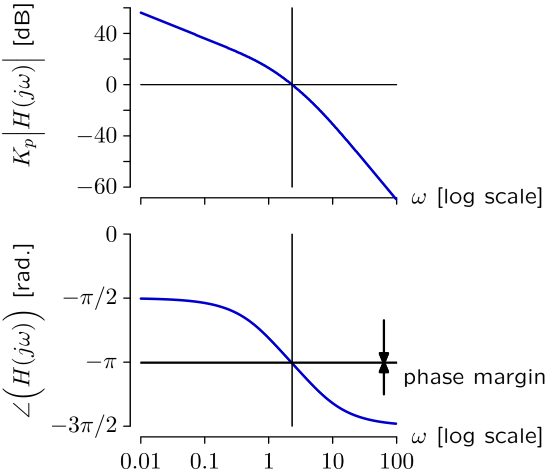


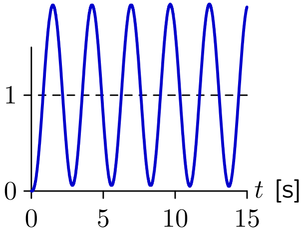
closed-loop step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 32$



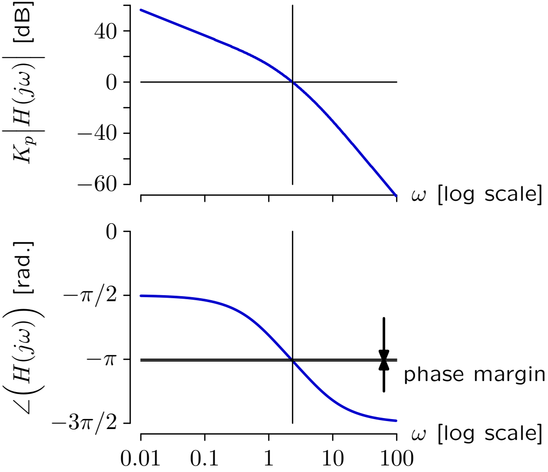


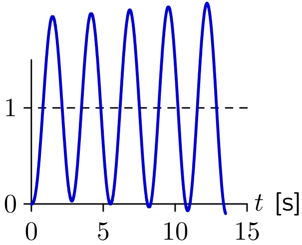
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Let ω_0 represent the frequency where $|H(j\omega_0)| = 1$. The system will be stable if the angle of $H(j\omega_0)$ is greater than $-\pi$ and unstable otherwise.

$K_p = 33$



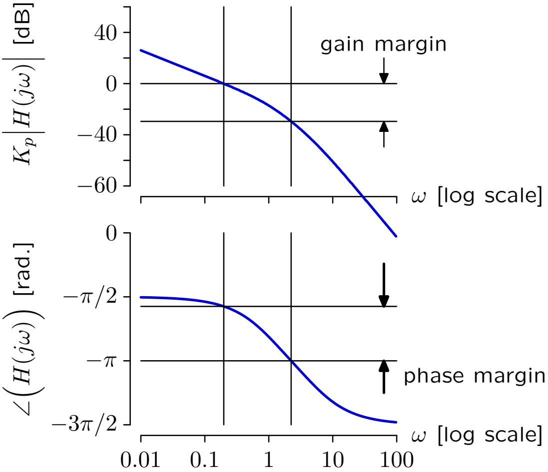


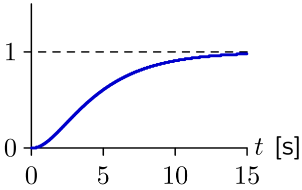
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 1$



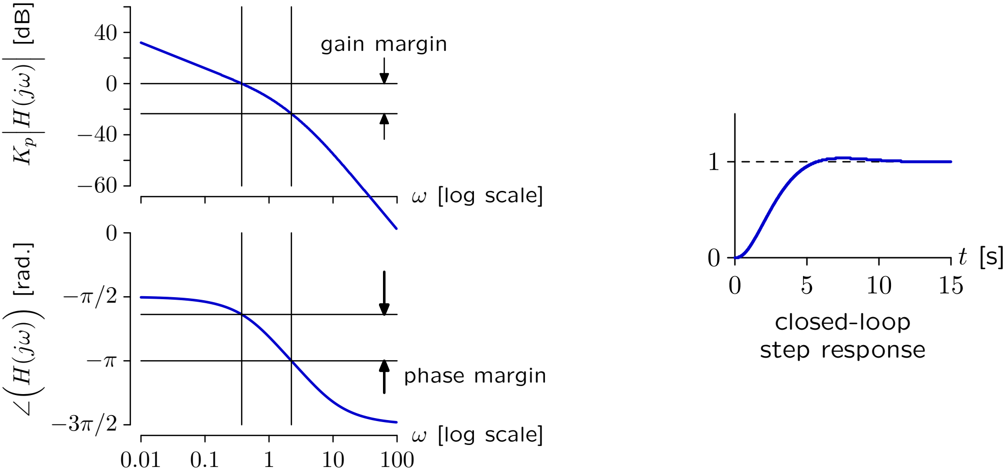


closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

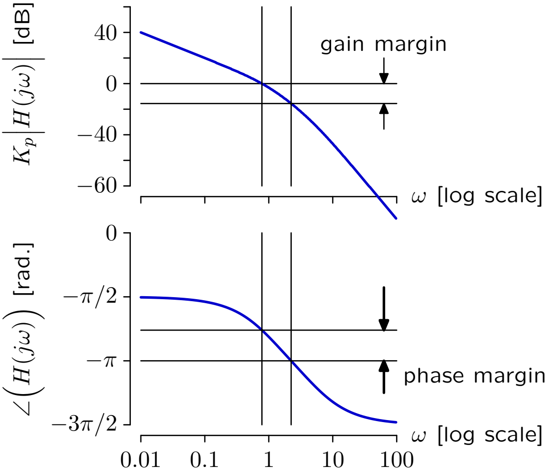
$K_p = 2$

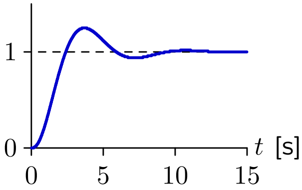


Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 5$



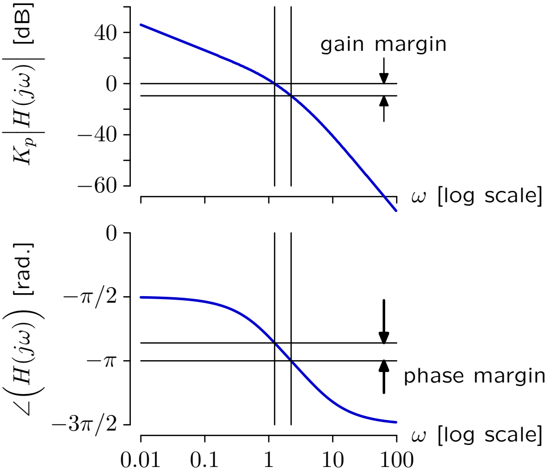


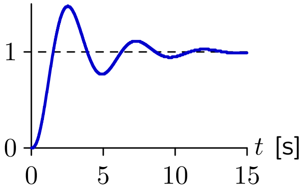
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 10$



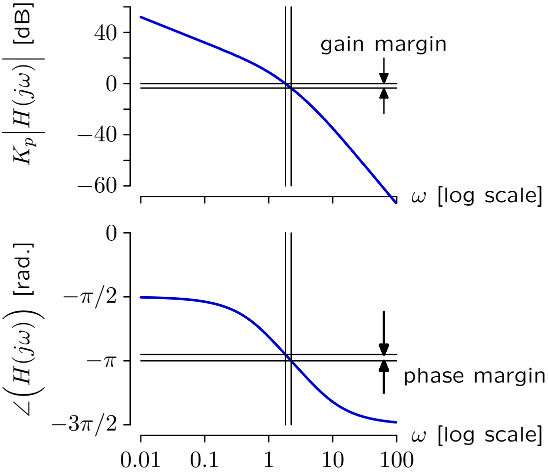


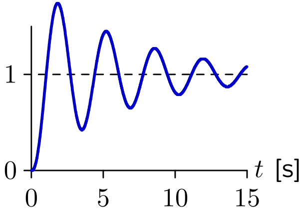
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 20$



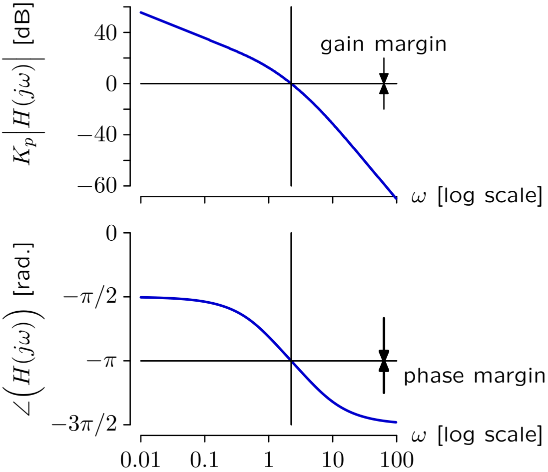


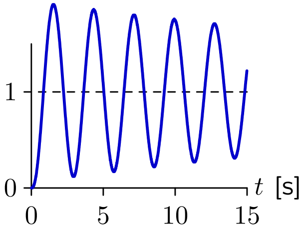
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 30$



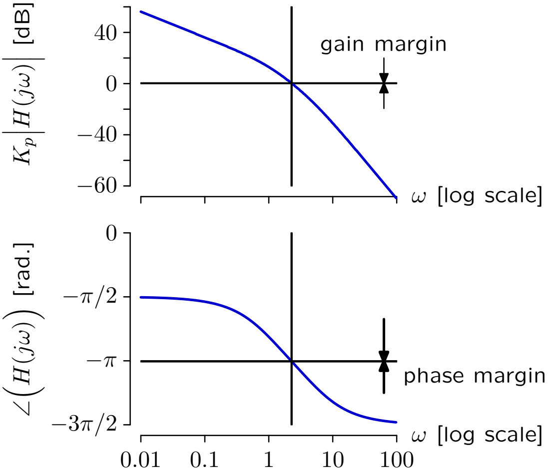


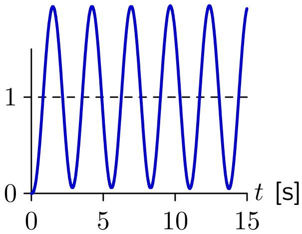
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 32$



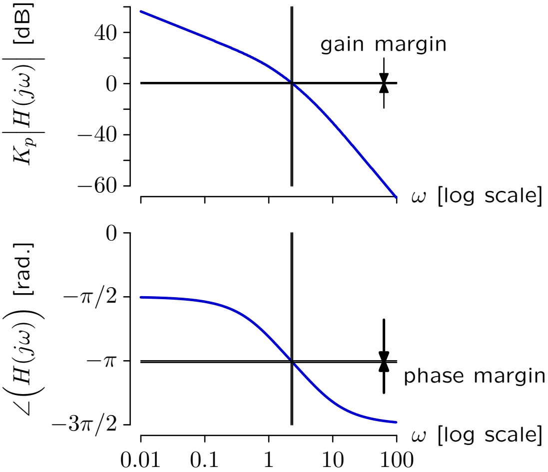


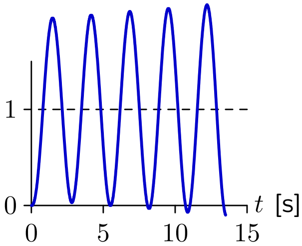
closed-loop
step response

Determining Stability from Open-Loop Frequency Response

Gain and phase margins provide useful stability metrics that can be computed directly from the open-loop frequency response.

$K_p = 33$

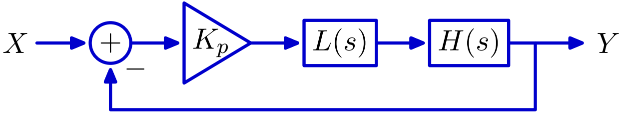




closed-loop
step response

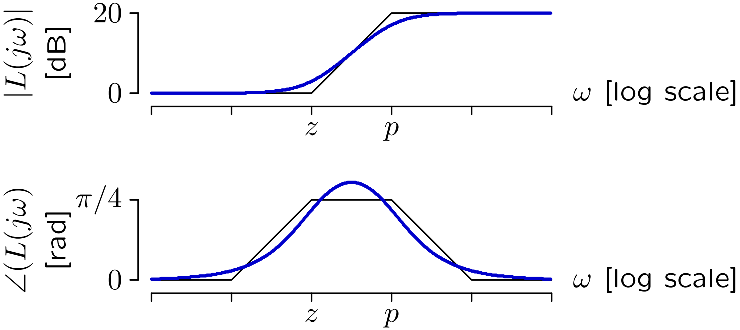
Lead Compensation

Stability can be enhanced by increasing the gain and/or phase margin using a **compensator** as shown below.



We can use a **lead** compensator to increase the phase margin.

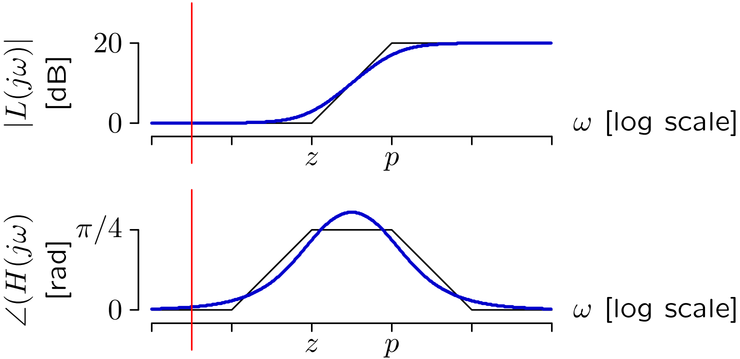
$$L(s) = \left(\frac{p}{z}\right)\left(\frac{s+z}{s+p}\right)$$



Lead Compensation

A lead compensator has no effect on the magnitude or phase at low frequencies.

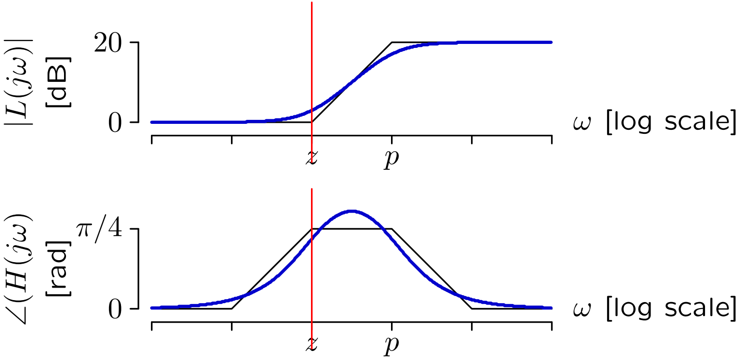
$$L(s) = \left(\frac{p}{z}\right)\left(\frac{s+z}{s+p}\right)$$



Lead Compensation

A lead compensator can significantly increase phase margin (which is good). Unfortunately, it also reduces the gain margin a bit (which is not so good).

$$L(s) = \left(\frac{p}{z}\right)\left(\frac{s+z}{s+p}\right)$$



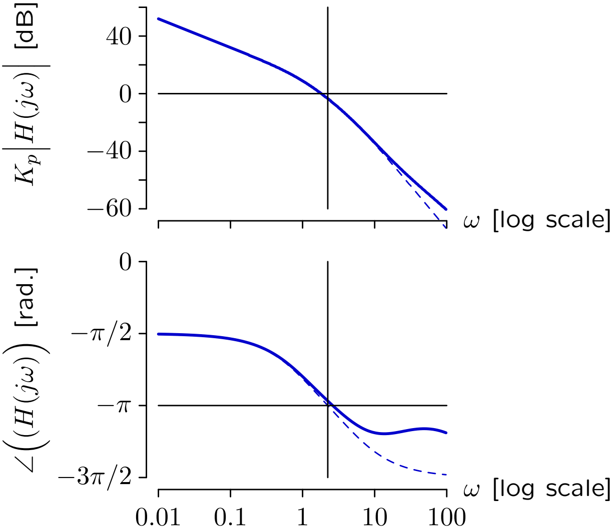
When adjusted appropriately, the increase in phase margin can more than compensate for the slight loss of gain margin.

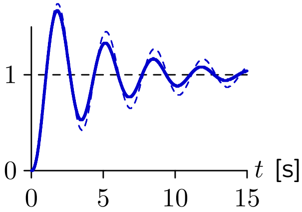
Improving Performance with Lead Compensation

Using a lead compensator with $z = 20$ and $p = 200$ has a very small effect.

$K_p = 20$

$z = 20; \quad p = 200$





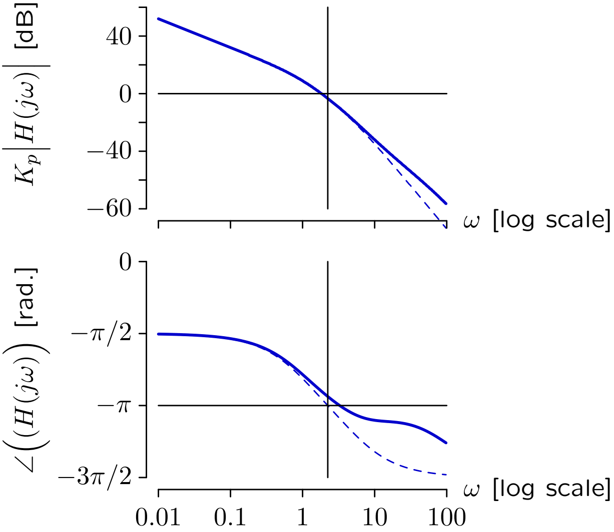
closed-loop
step response

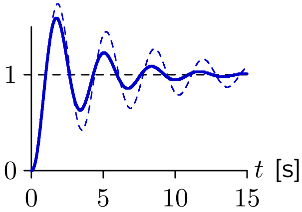
Improving Performance with Lead Compensation

Moving the compensator to a lower frequency increases convergence rate.

$K_p = 20$

$z = 10; \quad p = 100$





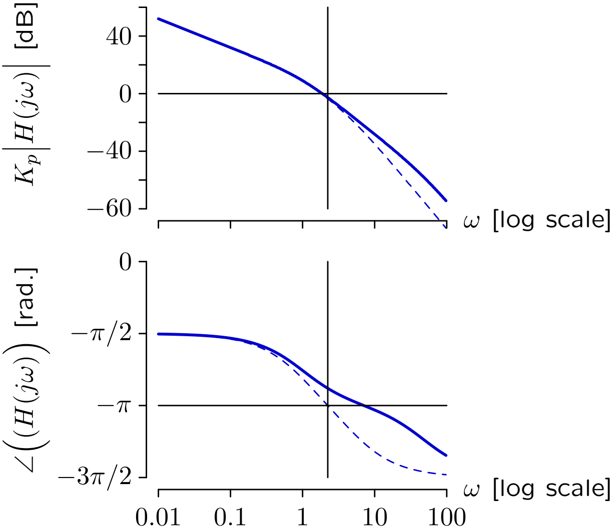
closed-loop
step response

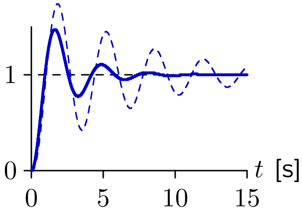
Improving Performance with Lead Compensation

Moving the compensator to a lower frequency increases convergence rate.

$K_p = 20$

$z = 5; \quad p = 50$





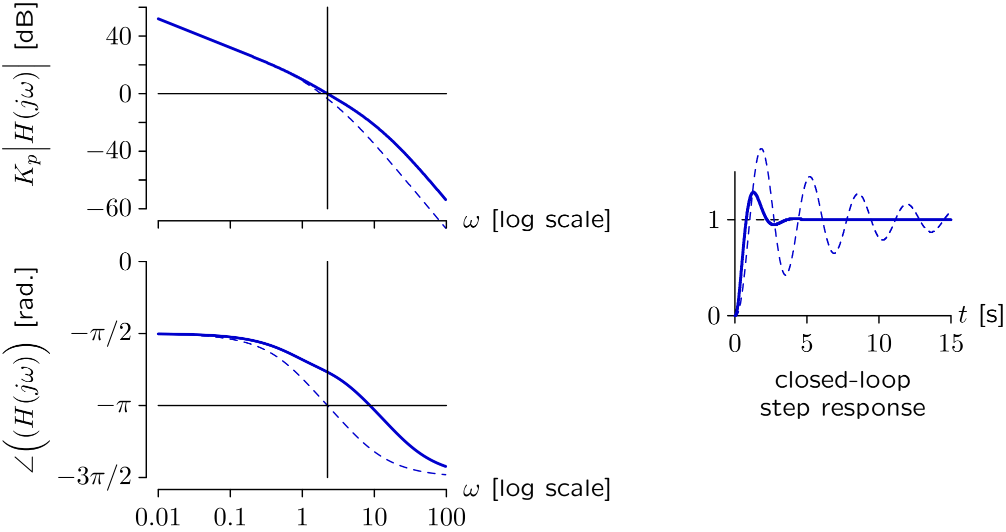
closed-loop
step response

Improving Performance with Lead Compensation

Convergence is dramatically improved when $z = 2$ and $p = 20$.

$K_p = 20$

$z = 2; \quad p = 20$

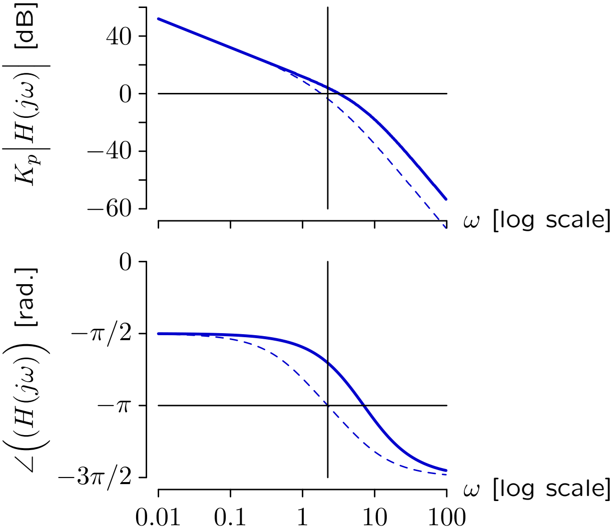


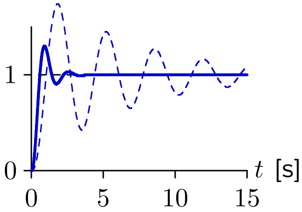
Improving Performance with Lead Compensation

Convergence for $z = 1$ not as good as $z = 2$ – now losing gain margin.

$K_p = 20$

$z = 1; \quad p = 10$





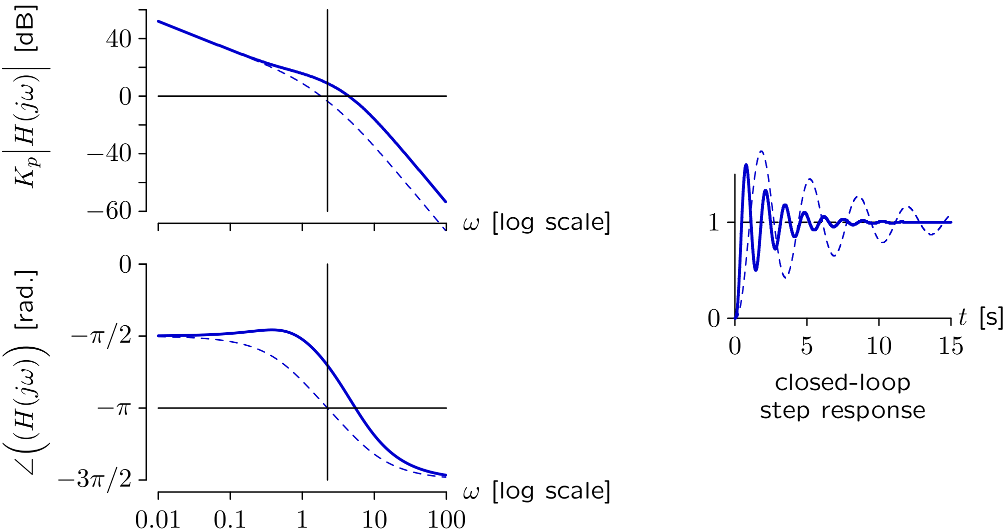
closed-loop
step response

Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.5$.

$K_p = 20$

$z = 0.5; \quad p = 5$



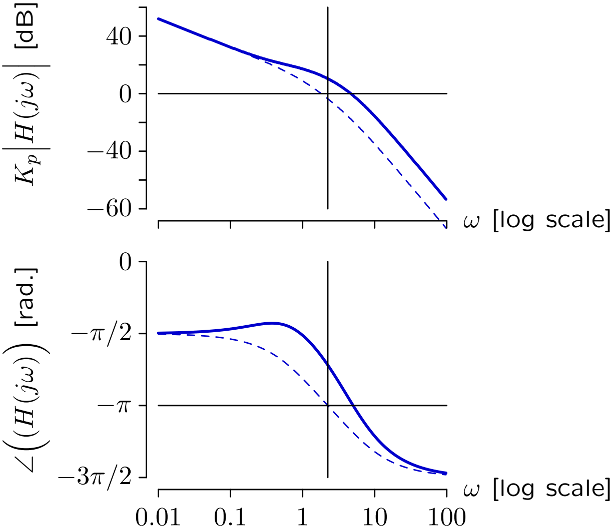
closed-loop
step response

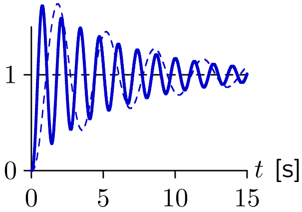
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.4$.

$K_p = 20$

$z = 0.4; \quad p = 4$





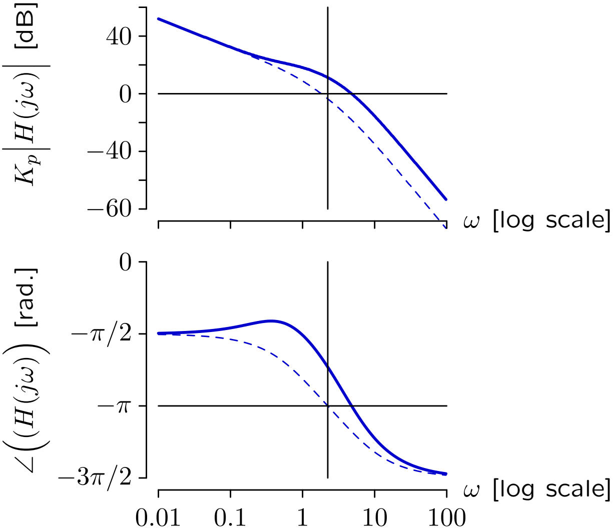
closed-loop
step response

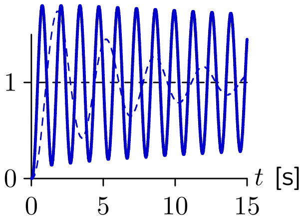
Improving Performance with Lead Compensation

The loss of gain margin is severe when $z = 0.35$.

$K_p = 20$

$z = 0.35; \quad p = 3.5$





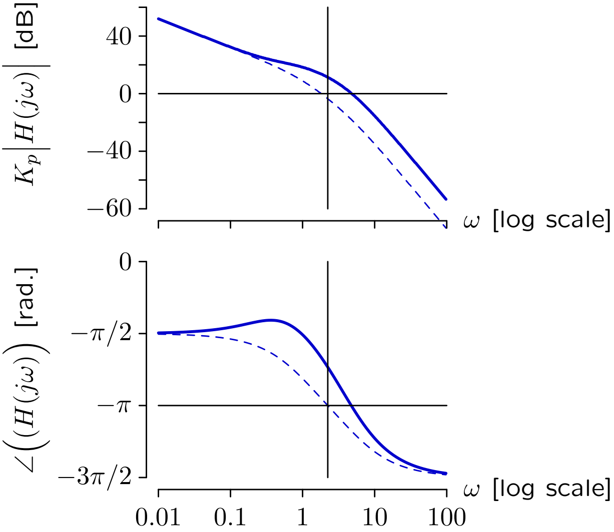
closed-loop
step response

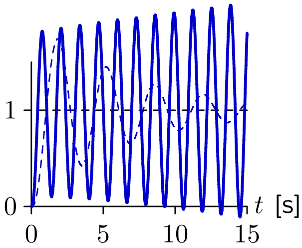
Improving Performance with Lead Compensation

The system is unstable when $z = 0.34$.

$K_p = 20$

$z = 0.34; \quad p = 3.4$





closed-loop
step response

Check Yourself

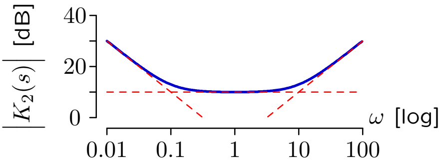
What is the relation (if any) between **lead compensation** and **PD control**?

Lead compensation and PD control are ...

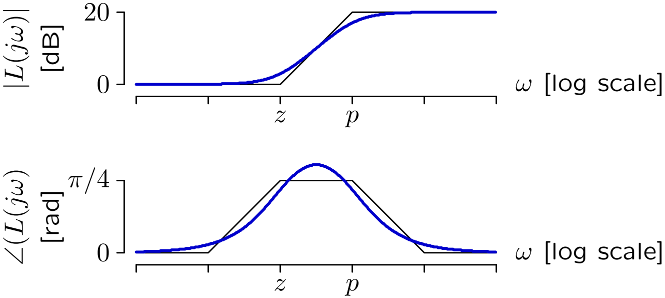
- equivalent if the zero in the lead compensator is at infinity
- equivalent if the pole in the lead compensator is at infinity
- equivalent if the zero in the lead compensator is at $-K_p/K_d$
- equivalent if the zero in the lead compensator is at $-K_p/K_d$ and the pole in the lead compensator is at infinity
- never equivalent

Check Yourself

PID control



lead compensation



Summary

Today we focused on a frequency-response approach to controller design.

Stability criterion: Let ω_0 represent the frequency at which the open-loop phase is $-\pi$. The closed loop system will be stable if the magnitude of the open-loop system at ω_0 is less than 1.

Useful metrics for characterizing relative stability:

- gain margin: ratio of the maximum stable gain to the current gain
- phase margin: additional phase lag needed to make system unstable

Lead compensation can improve performance by increasing phase margin (while also decreasing gain margin slightly).

Next time: root-locus method.