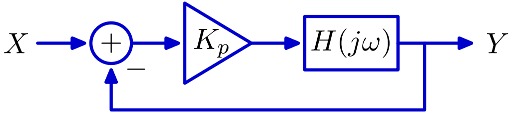
6.3100: Dynamic System Modeling and Control Design

Gain Margin, Phase Margin, and Root Locus

March 20, 2024

Last Time: Stability from Open-Loop Frequency Response

If $K_pH(j\omega_0) = -1$ then the closed-loop system has a pole at $s = j\omega_0$.



From Black's equation,

$$G(j\omega_0) = \frac{Y}{X} = \frac{K_pH(j\omega_0)}{1 + K_pH(j\omega_0)}$$

If $K_pH(j\omega_0) = -1$, then $|G(j\omega_0)| \to \infty$

But $G(s)$ can also be written as a ratio of first-order factors:

$$G(s) = K\frac{(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

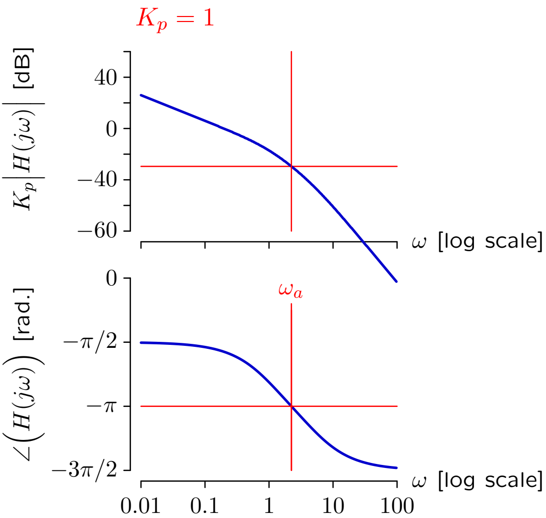
and if $G(j\omega_0) \to \infty$ then $j\omega_0$ is a root of the denominator.

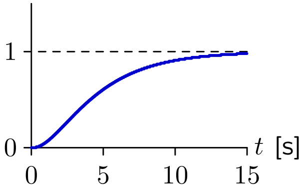
The closed-loop system $G(s)$ must have a pole at $s = j\omega_0$.

Gain Margin

Let ω_a represent the frequency where $\angle(H(j\omega_a)$ is $-\pi$.

The magnitude of $K_pH(j\omega_a)$ is < 1, so the closed-loop system is stable.



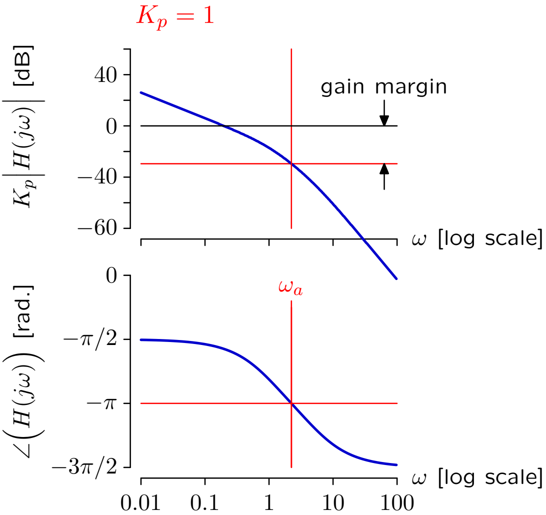


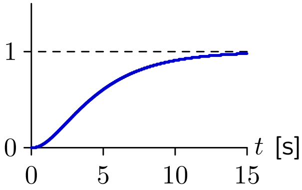
closed-loop
step response

Gain Margin

Let ω_a represent the frequency where $\angle(H(j\omega_a)$ is $-\pi$.

The gain margin is about 32 dB.





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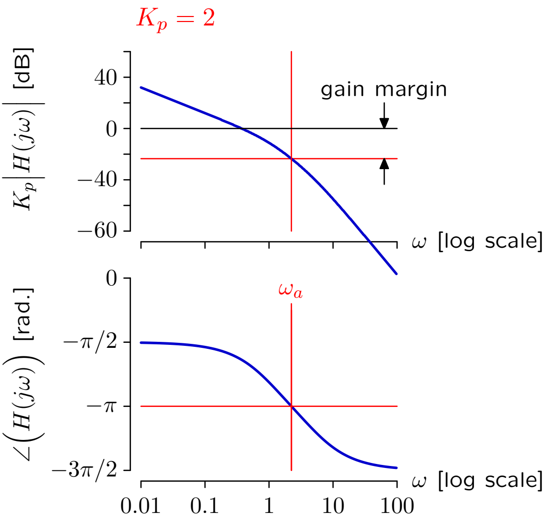
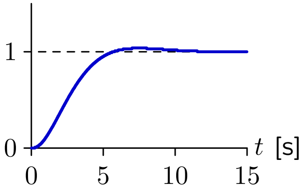
Let ω_a represent the frequency where $\angle(H(j\omega_a)$ is $-\pi$.

As $K_p \uparrow$ the gain margin shrinks and the step response becomes oscillatory.





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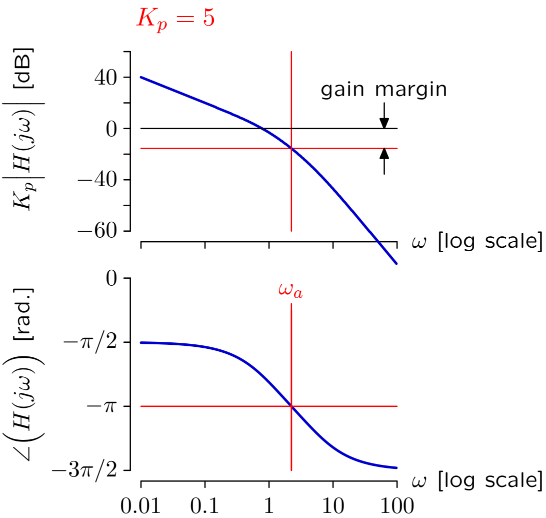

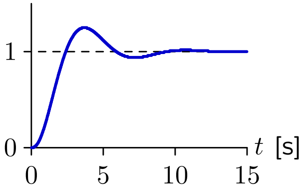

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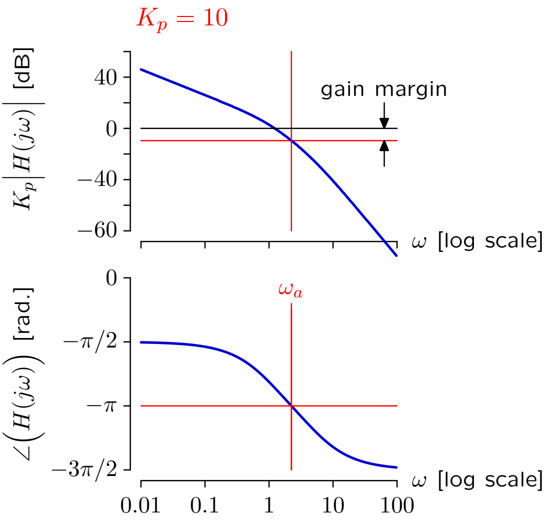

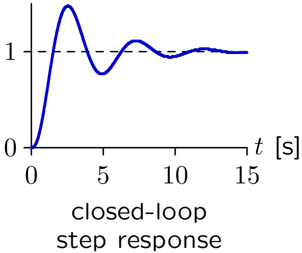

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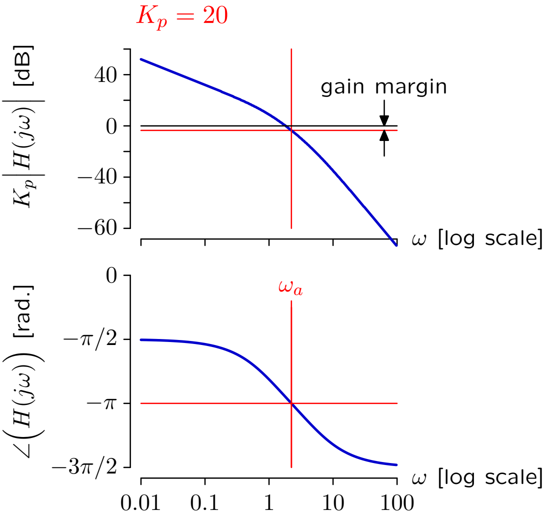

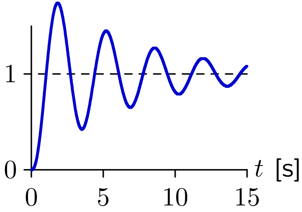

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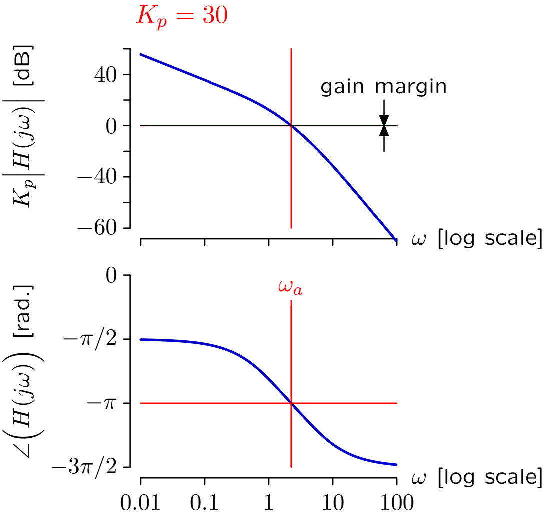

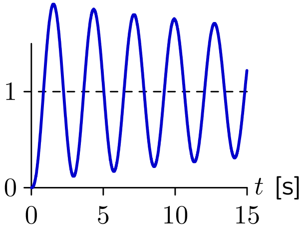

closed-loop
step response

Gain Margin

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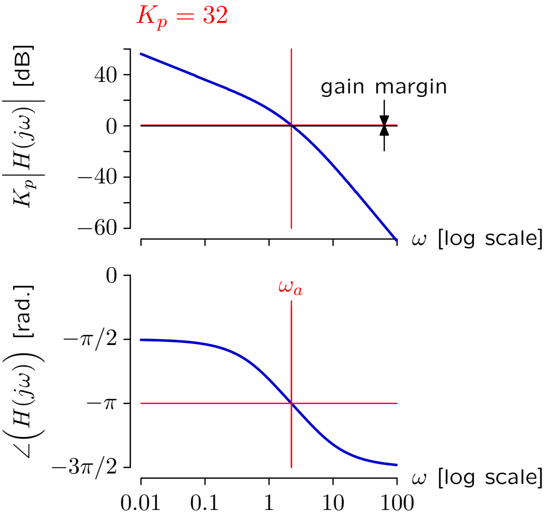
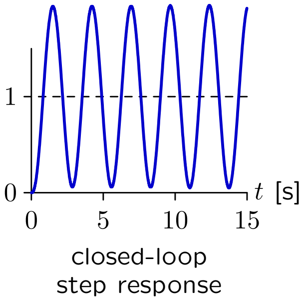
When gain margin $\to 0$, the closed-loop response no longer converges.

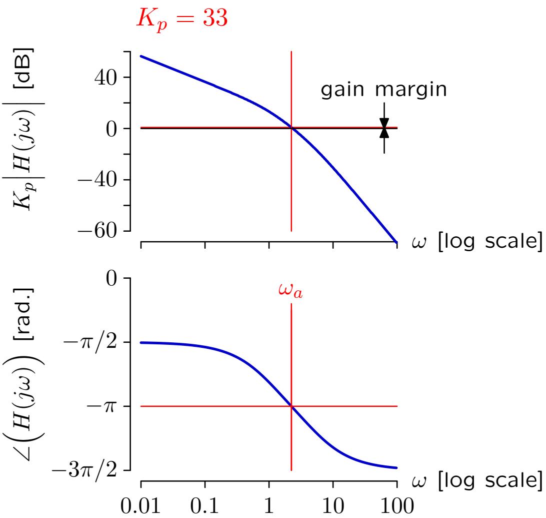


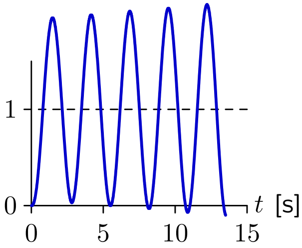


Gain Margin

Let ω_a represent the frequency where $\angle(H(j\omega_a)$ is $-\pi$.

When the gain margin goes negative, the closed-loop system is unstable.



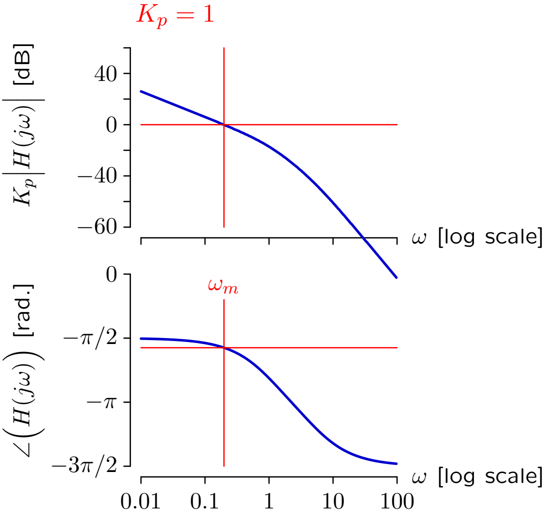


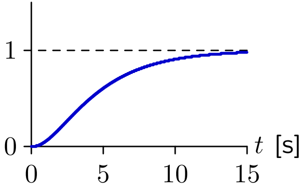
closed-loop
step response

Phase Margin

Let ω_m represent the frequency where $|K_p H(j\omega_m)| = 1$.

The angle of $H(j\omega_m)$ is greater than $-\pi$ so the closed-loop system is stable.



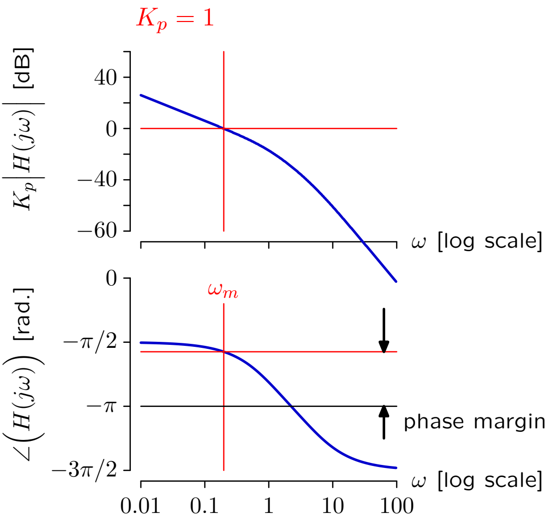


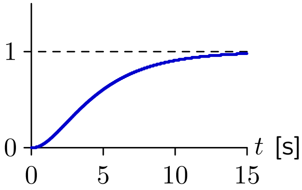
closed-loop
step response

Phase Margin

Let ω_m represent the frequency where $|K_p H(j\omega_m)| = 1$.

The phase margin is almost $\pi/2$.



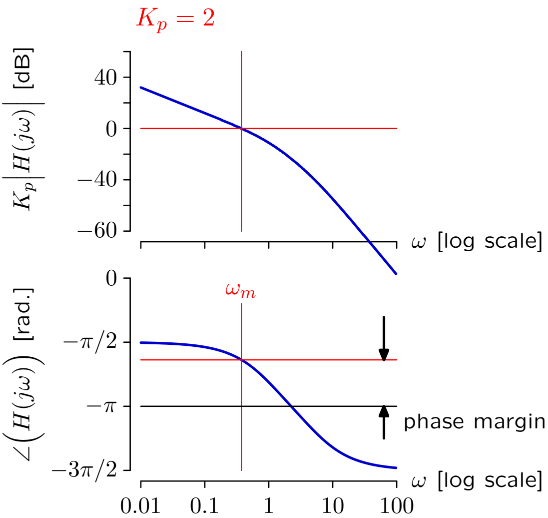


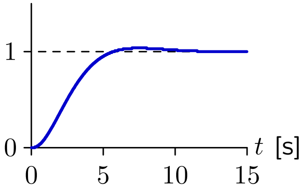
closed-loop
step response

Phase Margin

Let ω_m represent the frequency where $|K_pH(j\omega_m)| = 1$.

As $K_p \uparrow$ phase margin shrinks and step response becomes oscillatory.





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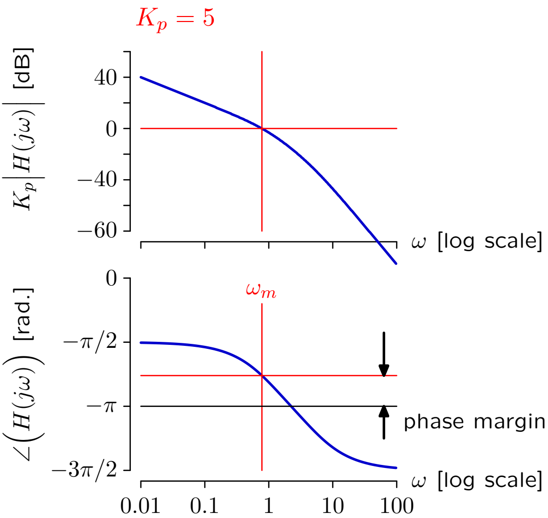

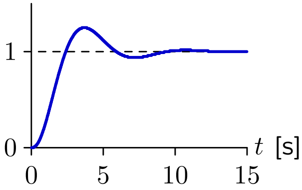

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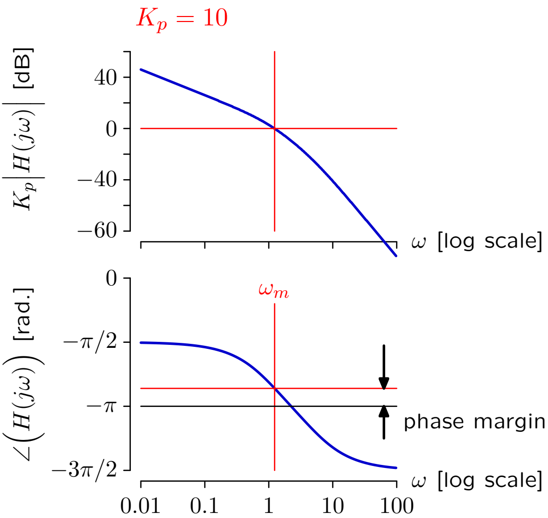

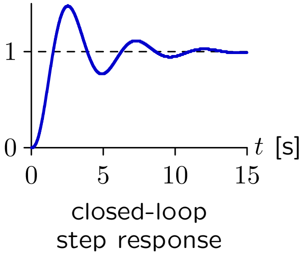

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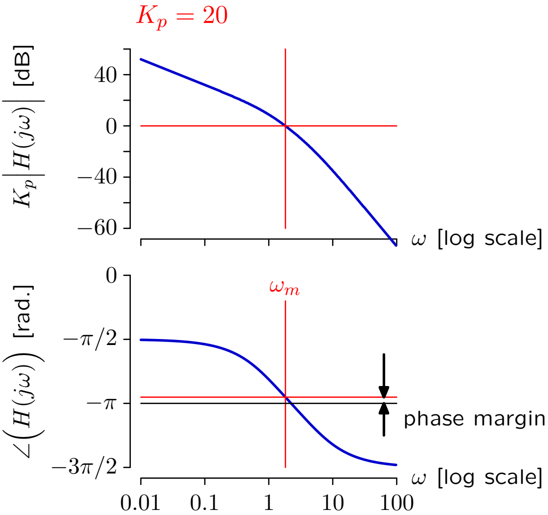

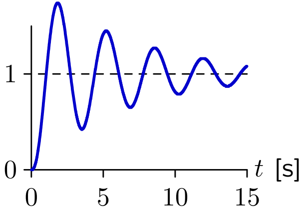

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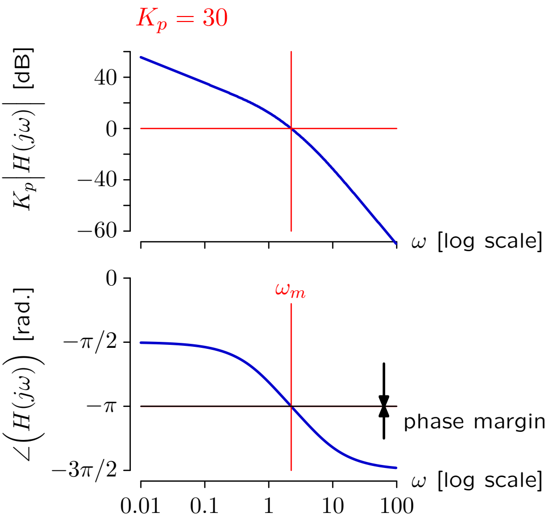

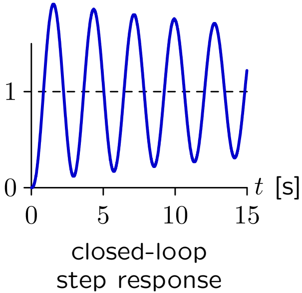

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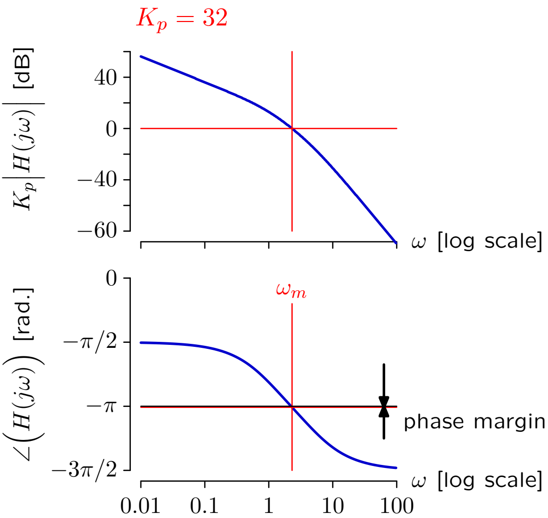

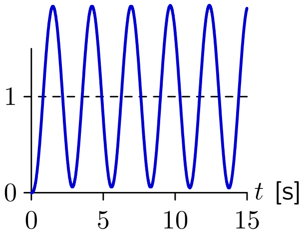

closed-loop
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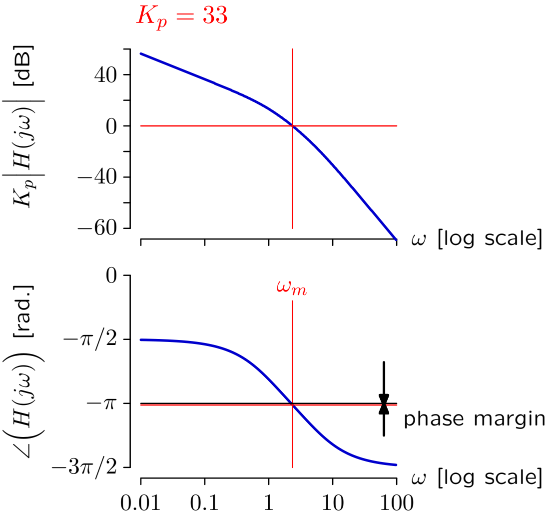
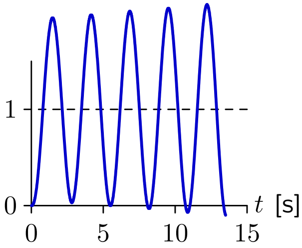
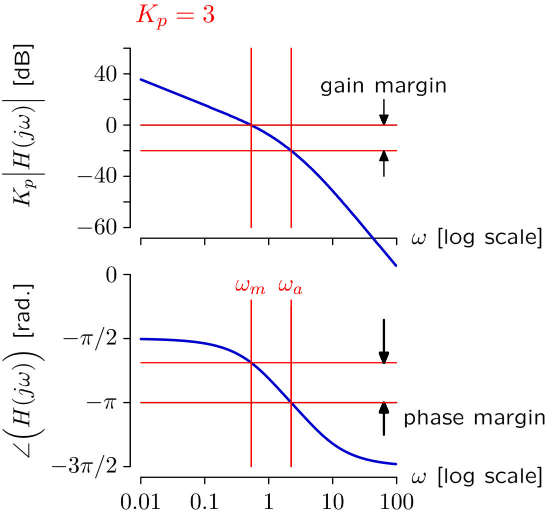


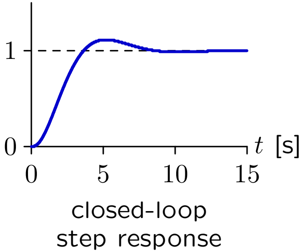


closed-loop
step response

Two New Metrics: Gain Margin and Phase Margin

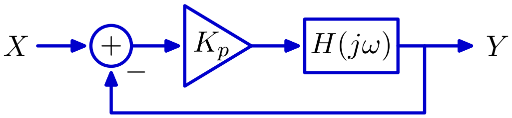
We would typically specify some minimum gain margin **and** some minimum phase margin.





From the Imaginary Axis ...

The closed-loop system will have a zero at $s = j\omega_0$ if $K_p H(j\omega_0) = -1$.



From Black's equation,

$$G(j\omega_0) = \frac{Y}{X} = \frac{K_p H(j\omega_0)}{1 + K_p H(j\omega_0)}$$

If $K_p H(j\omega_0) = -1$, then $|G(j\omega_0)| \to \infty$

But $G(s)$ can also be written as a ratio of first-order factors:

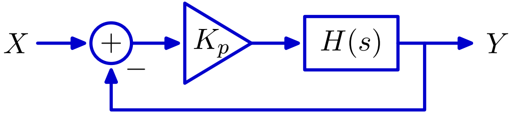
$$G(s) = K \frac{(s - z_1)(s - z_2)(s - z_3)\cdots}{(s - p_1)(s - p_2)(s - p_3)\cdots}$$

and if $G(j\omega_0) \to \infty$ then $j\omega_0$ is a root of the denominator.

The closed-loop system $G(s)$ must have a pole at $s = j\omega_0$.

... to the Entire Complex Plane

The closed-loop system will have a zero at $s{=}s_0$ if $K_pH(s_0){=}-1$.



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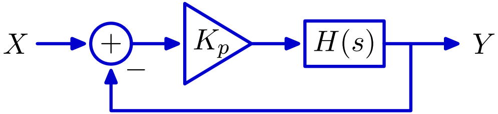
$$G(s) = K\frac{(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

and if $G(s) \to \infty$ then s_0 is a root of the denominator.

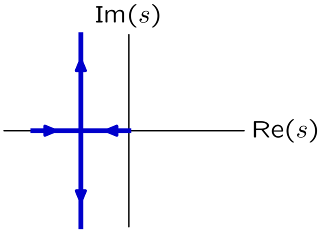
The closed-loop system $G(s)$ must have a pole at $s = s_0$.
The collection of all such s_0 is called a **root locus**.

Root Locus

A **root locus** shows points in the s-plane that are poles of the closed loop system function $G(s) = Y/X$ for values of $K_p > 0$.



Example: Root locus for $H(s) = \dfrac{1}{s(s+1)}$



Given an expression for $H(s)$, we can easily calculate the poles of the closed-loop system function $G(s)$ numerically.

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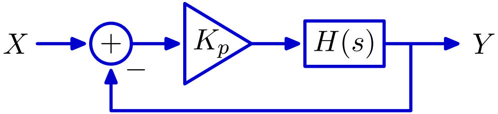

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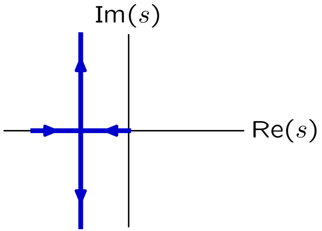
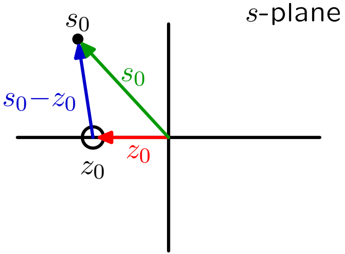


A more intuitive (and often more informative) method is to solve the stability criteria using vectors to represent the open-loop transfer function $H(s)$.

Vector Analysis

The **transfer function** of a system composed of adders, gains, differentiators, and integrators can be determined from **vectors** associated with the system's poles/zeros.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$



Combine the vector representation with the stability criteria:

- $\left|K_pH(s_0)\right| = 1$ and
- $\angle(K_pH(s_0) = -\pi\ (\pm k2\pi)$

$\Bigg\}\ K_pH(s_0) = -1$

to find the root locus.

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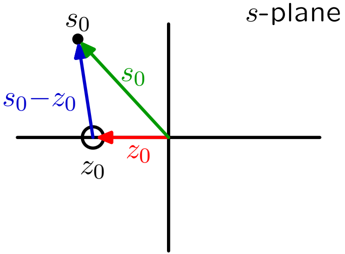

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$\Big\}\; K_pH(s_0) = -1$

Surprisingly, the **angle relation** is easiest to work with.

Root Locus

The shape of the root locus follows from a few simple rules.

$$G(s) = \frac{K_p H(s)}{1 + K_p H(s)}$$

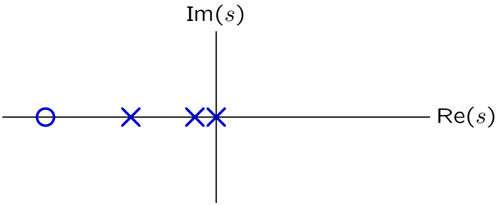
Starting Rule: Each root locus branch starts at an **open-loop pole**.

For $0 < K_p << 1$, the denominator of $G(s) \to 1$ and

$$G(s) \to K_p H(s)$$

The closed-loop poles of $G(s)$ are equal to the open-loop poles of $H(s)$.

Example: The following plot shows open-loop poles/zeros of a plant $H(s)$:

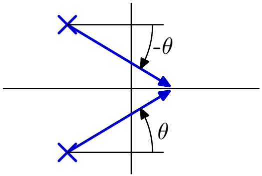


The associated root locus has 3 branches, one starting from each pole.

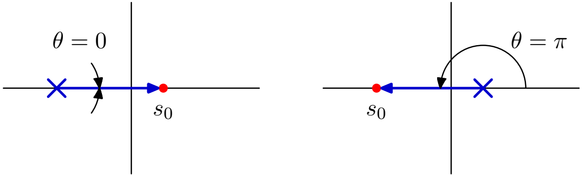
Root Locus

Real-Axis Rule: A point on the real axis is in the root locus if # of poles to the right of the point plus # of zeros to the right of the point is **odd**.

If a system contains just adders, gains, differentiators, and integrators, then poles (and zeros) with nonzero imaginary parts come in conjugate pairs, and do not contribute to the angle of $H(s)$ if s is on the real axis.



A real-valued pole or zero contributes 0 or π to the angle of $H(s_0)$ depending on whether s_0 is to the right or left of the pole or zero.



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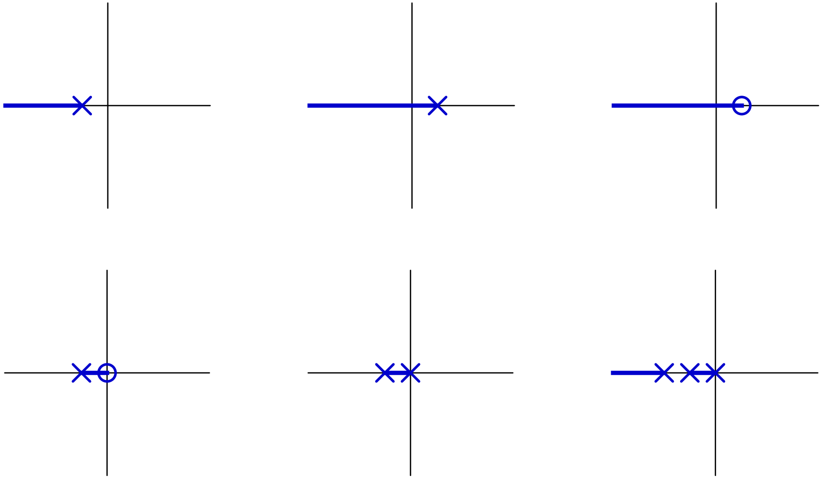
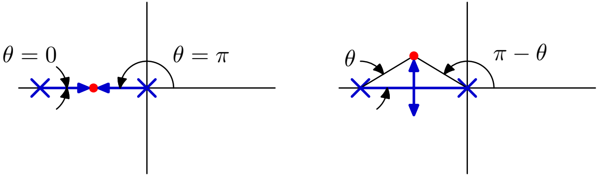
Examples:



Root Locus

Break-Away Rule: Increasing K_p after two real-valued closed-loop poles collide causes them to split off the real axis.

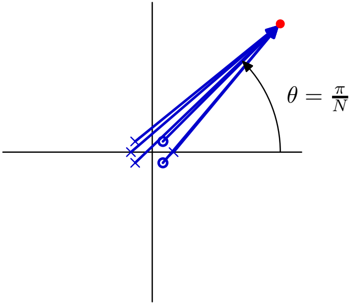
The left panel below shows two real-valued, closed-loop poles approaching each other. Notice that their angles sum to π prior to collision.
The right panel below shows that the angles still sum to π after the collision.



Root Locus

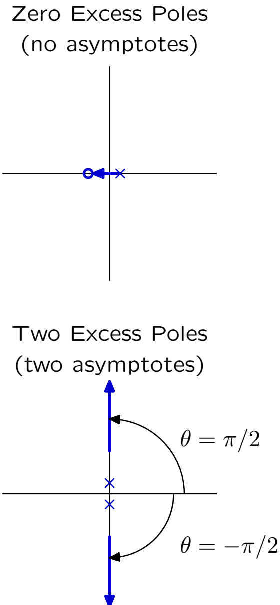
High-Gain Rule: If the # of poles exceeds the # of zeros by $N>0$, there will be N high-gain asymptotes with angles at odd multiples of π/N.

When $|s|$ is large, vectors from the poles and zeros of $H(s)$ to s will be approximately equal. Since the angle from a pole will be equal to the angle from a zero, the angles from pole/zero pairs will cancel, leaving a net number of excess poles (N) whose angles must sum to π.



Root Locus

High-Gain Rule: If the # of poles exceeds the # of zeros by N, there will be N high-gain asymptotes with angles at $(2n+1)\pi/N$.



Root Locus

Mean Rule: If # of poles is at least two greater than the # of zeros, then the average closed-loop pole position is independent of K_p.

Example:

$$H(s) = \frac{s+z}{(s+p_1)(s+p_2)(s+p_3)}$$

$$G(s) = \frac{\frac{s+z}{(s+p_1)(s+p_2)(s+p_3)}}{1 + \frac{K_p(s+z)}{(s+p_1)(s+p_2)(s+p_3)}}$$

$$= \frac{s+z}{(s+p_1)(s+p_2)(s+p_3) + K_p(s+z)}$$

$$= \frac{s+z}{s^3 + (p_1+p_2+p_3)s^2 + (p_1p_2+p_1p_3+p_2p_3)s + (p_1p_2p_3) + K_ps+K_pz}$$

$$= \frac{s+z}{s^3 + (p_1+p_2+p_3)s^2 + (p_1p_2+p_1p_3+p_2p_3 + K_p)s + (p_1p_2p_3 + K_pz)}$$

The sum of the closed-loop poles $(p_1+p_2+p_3)$ does not depend on K_p.

Root Locus

Ending Rule: Each root locus branch **ends at an open-loop zero or ∞.**

As $K_p \to \infty$, $|H(s)|$ must approach 0 to satisfy the magnitude criterion $|K_p H(s)| = 1$.

If the number of open-loop zeros (n_z) is greater than or equal to the number of open-loop poles (n_p), each branch of the root locus will end at an open-loop zero.

If n_z is less than n_p, then $n_p - n_z$ branches must go to infinity. As $|s| \to \infty$,

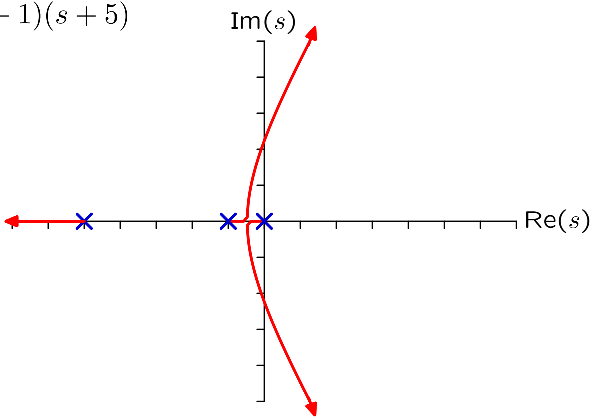
$$H(s) = K\frac{(s-z_1)(s-z_2)(s-z_3)\cdots(s-z_{n_z})}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_{n_p})}$$

will approach zero since the order of the denominator is greater than that of the numerator.

Example: Root Locus Analysis

Root locus for the problem from the beginning of lecture.

$$H(s) = \frac{1}{s(s+1)(s+5)}$$



$K_p = 0$: three real-valued poles (two dominant).

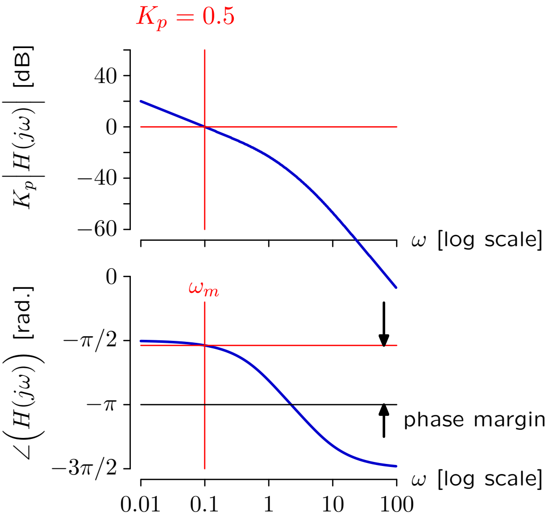
$0 < K_p < 1$: real poles at $s=0$ and -1 move toward each other.

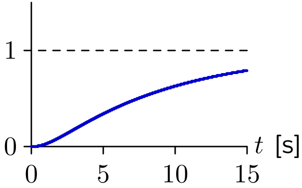
$1 < K_p < 32$: complex poles $\rightarrow$ oscillations increase in freq and persistence.

$K_p > 32$: complex pole-pair goes unstable.

Example: Frequency Response Analysis

If $0<K_p<1$ there are two real-valued poles.

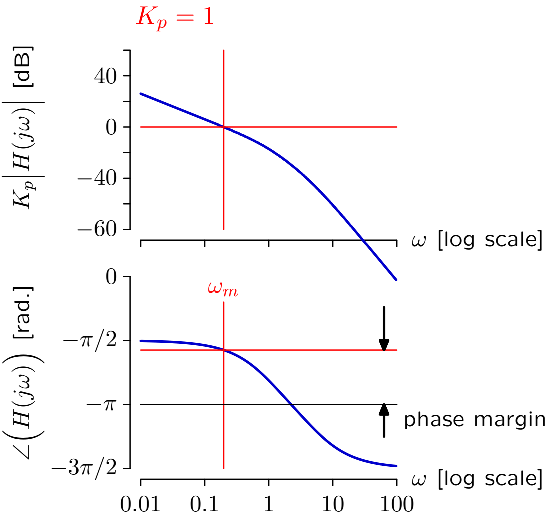


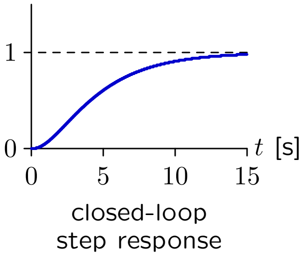


closed-loop
step response

Example: Frequency Response Analysis

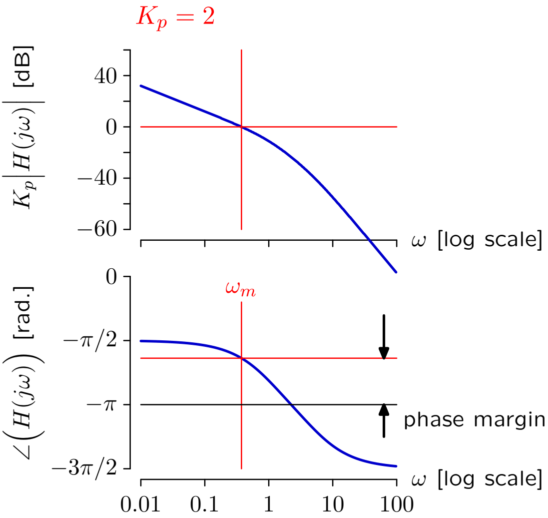
If $0 < K_p < 1$ there are two real-valued poles.

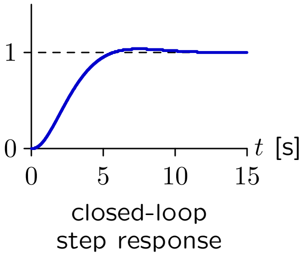




Example: Frequency Response Analysis

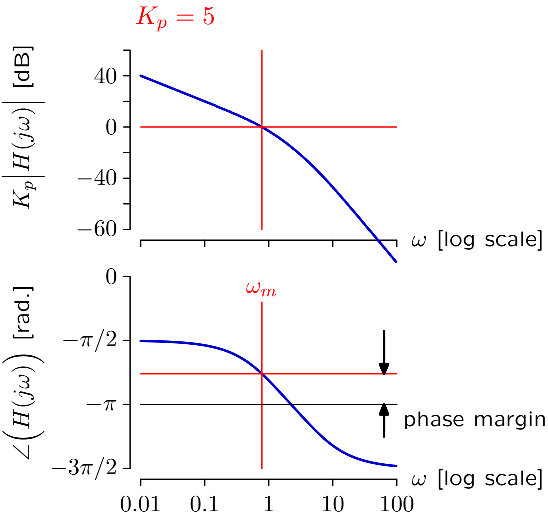
If $1 < K_p < 32$ oscillation increases in frequency and persistence.

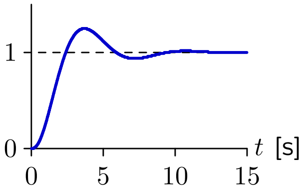




Example: Frequency Response Analysis

If $1 < K_p < 32$ oscillation increases in frequency and persistence.





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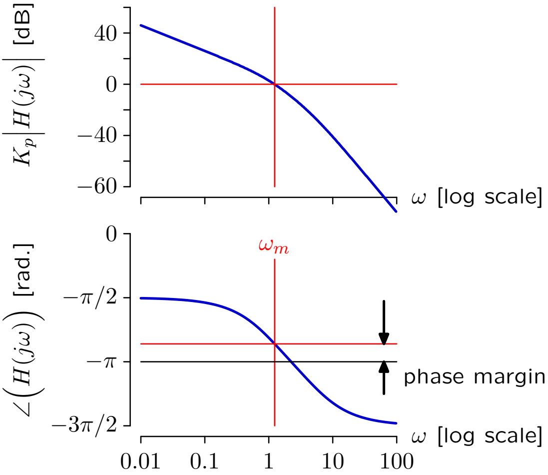
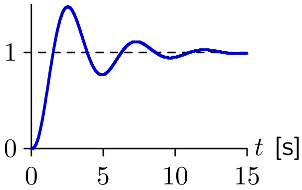
step response

Example: Frequency Response Analysis

If $1 < K_p < 32$ oscillation increases in frequency and persistence.

$K_p = 10$





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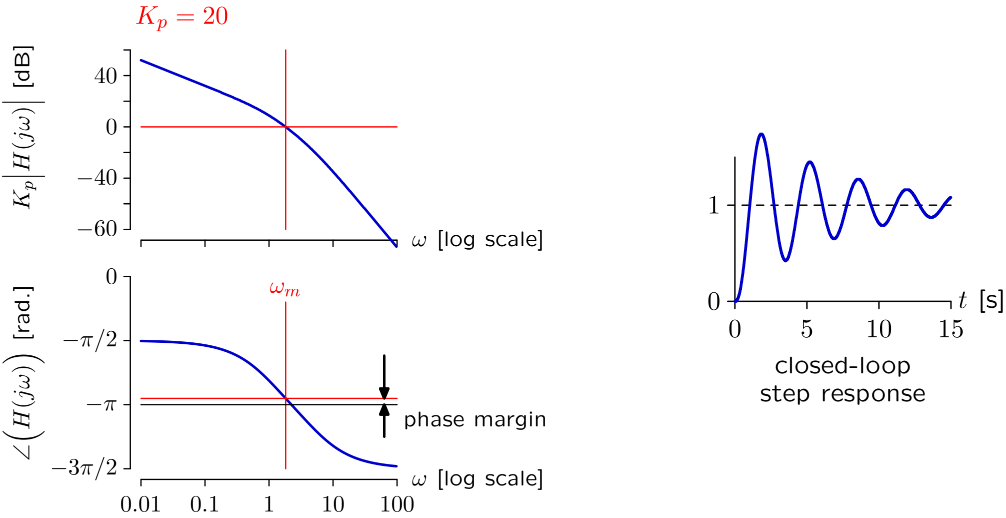
step response

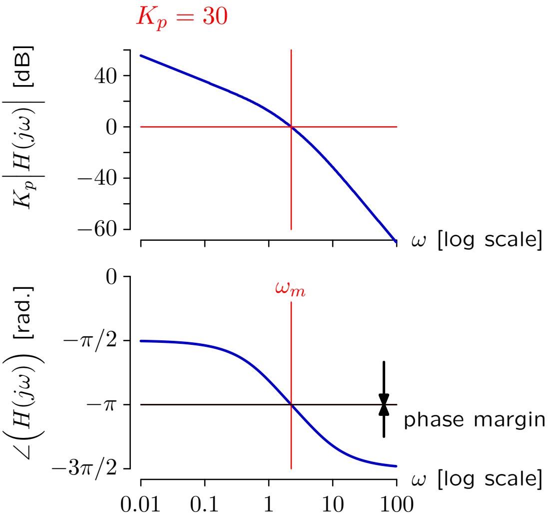
Example: Frequency Response Analysis

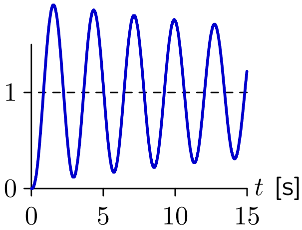
If $1 < K_p < 32$ oscillation increases in frequency and persistence.



Example: Frequency Response Analysis

If $1 < K_p < 32$ oscillation increases in frequency and persistence.





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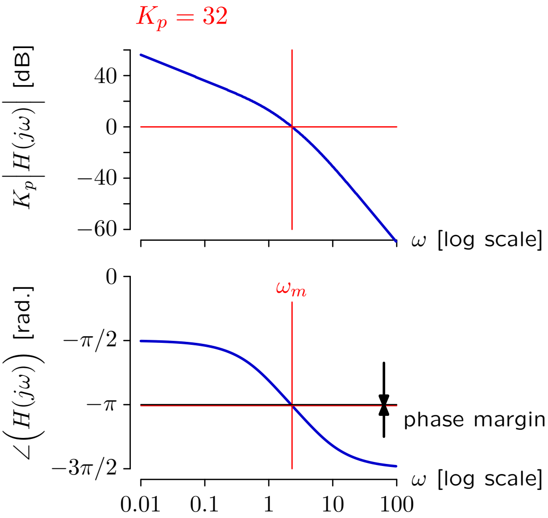
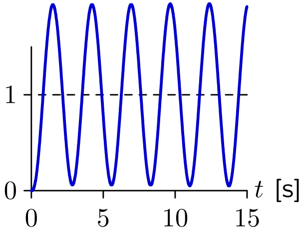
Example: Frequency Response Analysis

If K_p=32 persistent oscillation





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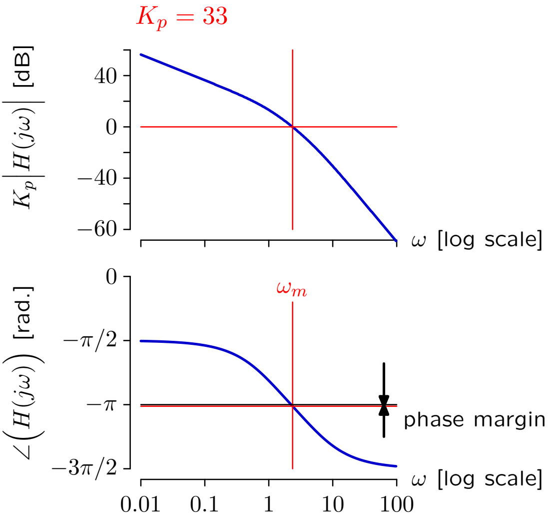
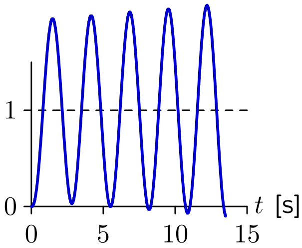
step response

Example: Frequency Response Analysis

If $K_p > 32$ unstable.





closed-loop
step response

Example: Root Locus Analysis

Return to problem from beginning of lecture:

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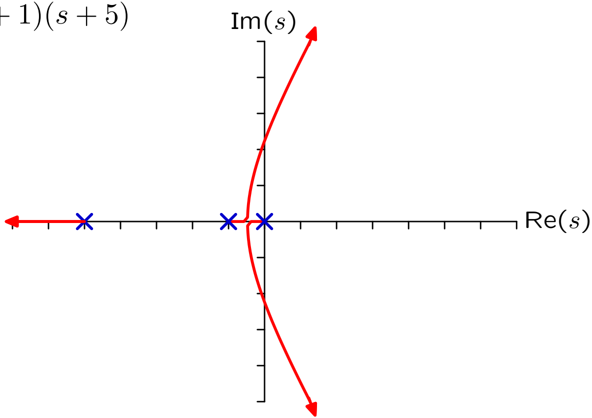

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$K_p > 32$: complex pole-pair goes unstable.

Summary

Today we focused on the root-locus method to analyze and design controllers.

This method builds on the frequency response method from last lecture.

Both methods are based on the observation that the poles of a closed-loop system are at the frequencies s_0 where the open-loop system is -1.