6.3100: Dynamic System Modeling and Control Design

Controlling a System with an Observer

April 24, 2024

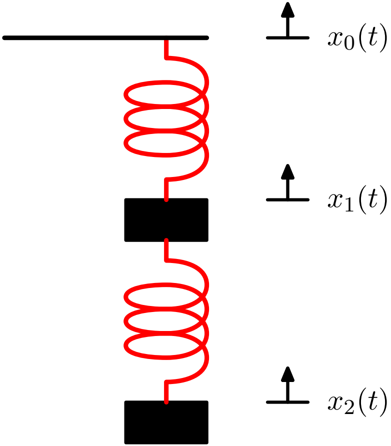
Controlling a System with an Observer

Today we will introduce a new method of control based on **observers**.

To see how this new method builds on previous ideas, let's consider all of these methods in the context of a particular problem.

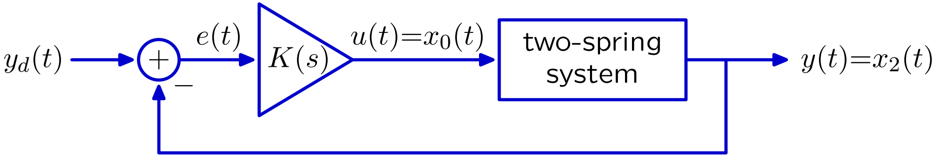
Two-Spring System

The **plant** consists of two springs and two masses. The goal is to move the input $u(t) = x_0(t)$ so as to move the bottom mass to a desired location $x_2(t) = y_d(t)$.



Classical Control

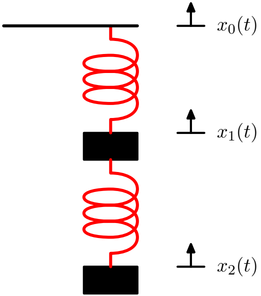
A classical controller for this problem has the following form.



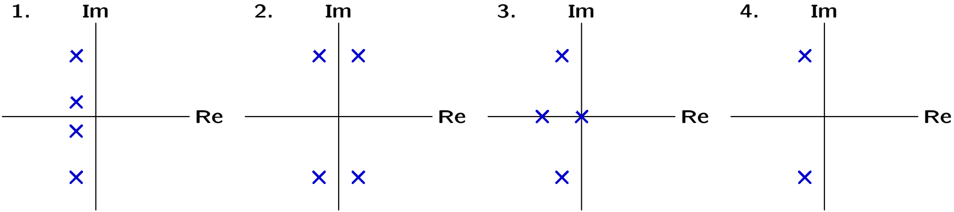
To solve this classical control problem, we must

- find the equations of motion for the plant (the two-spring system) and
- express those equations in terms of a transfer function.

Check Yourself

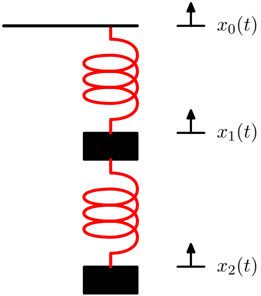


Which plot (if any) shows the poles of the two-spring system?



Two-Spring System

Equations of motion.



$$f_{m1} = m\ddot{x}_1(t) = k\Big(x_0(t) - x_1(t)\Big) - k\Big(x_1(t) - x_2(t)\Big) - b\dot{x}_1(t) - mg$$

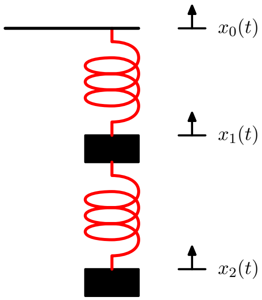
$$f_{m2} = m\ddot{x}_2(t) = k\Big(x_1(t) - x_2(t)\Big) - b\dot{x}_2(t) - mg$$

mass $m = 1$, stiffness $k = 2$, damping $b = 1.4$.

Positions $x_1(t)$ and $x_2(t)$ result from two separable inputs: gravity mg, which generates constant offsets, and $x_0(t)$, which determines the dynamics.

Two-Spring System

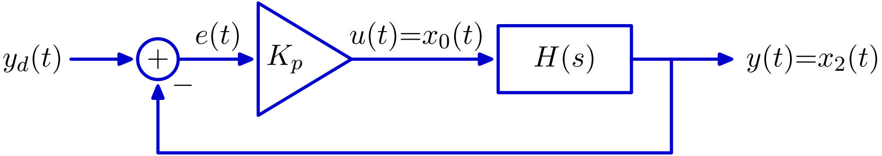
Transfer function.



$$H(s) = \frac{X_2(s)}{X_0(s)} = \frac{k^2}{(s^2m + sb + 2k)(s^2m + sb + k) - k^2}$$

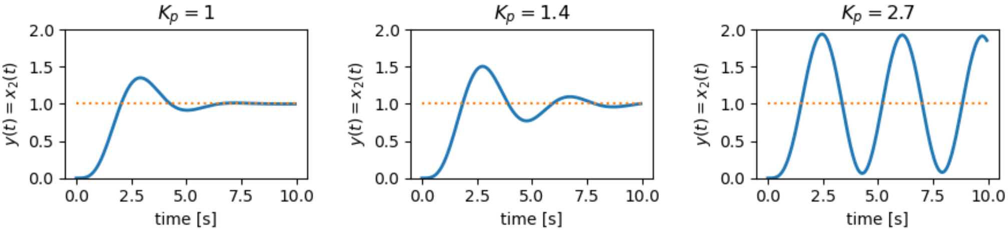
Classical Control

A proportional controller has the following form.



The feedback system is stable for only a small range of gains: $K_p < 2.7$

Step responses:

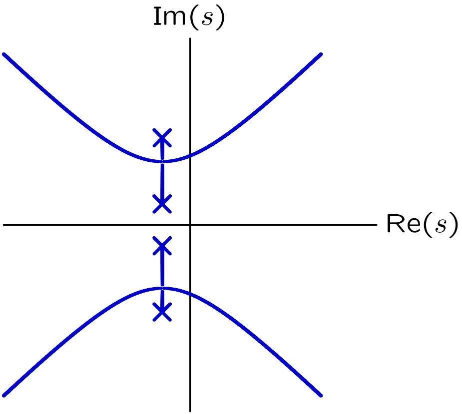


Slow convergence and large oscillatory overshoots.

Why such poor behavior?

Classical Control

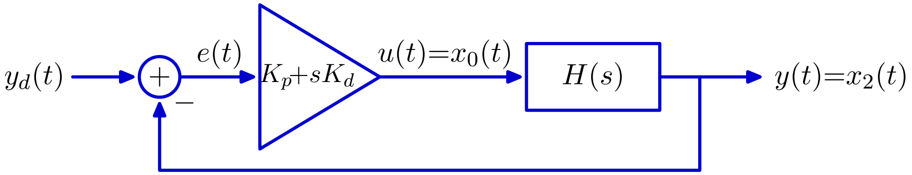
Root locus.



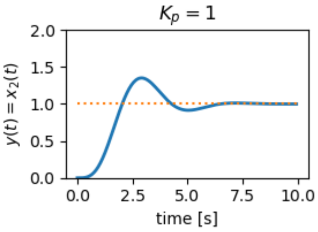
Track the open-loop poles as the gain K increases, starting at zero.

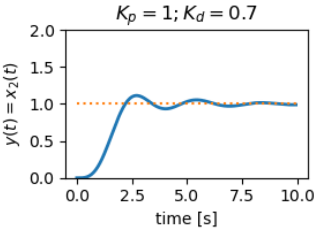
Classical Control

Proportional plus derivative performance is similar to that for proportional.



Step responses:

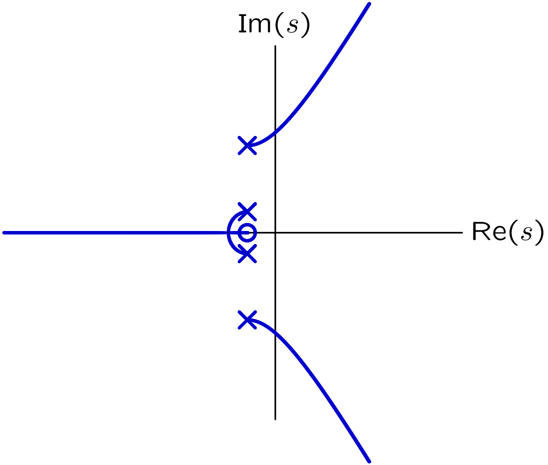




Smaller overshoot, but still slow convergence.

Classical Control

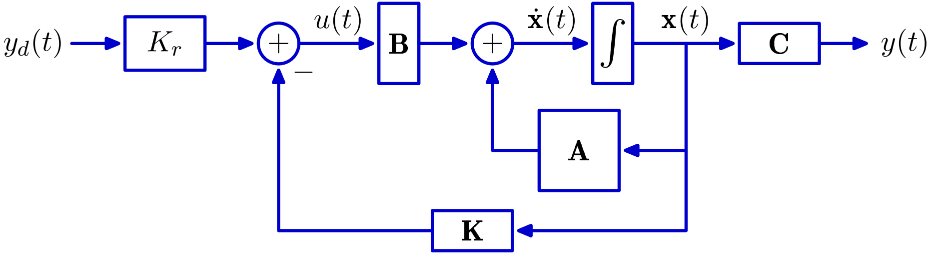
Root locus.

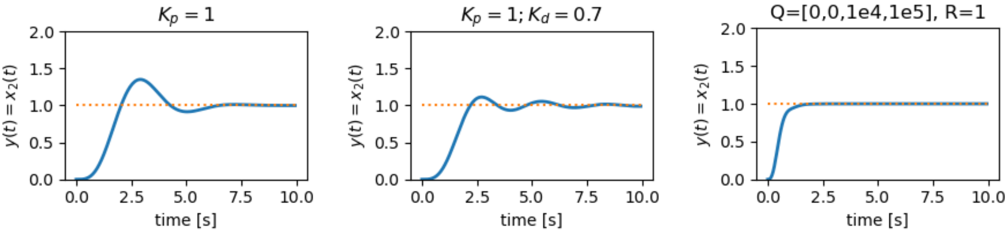


Track the open-loop poles as the gain K_p increases with $K_d = K_p/0.7$.

State-Space Control

State-space control is **much** better.



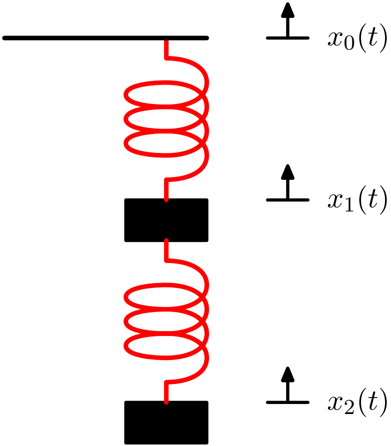


What is it about state-space control that allows better performance?

Two-Spring System

The state-space approach uses information from $x_2(t)$ **and** $x_1(t)$.

The combination of $x_1(t)$ and $x_2(t)$ is much more powerful than $x_2(t)$ alone.



Beyond State-Space Control

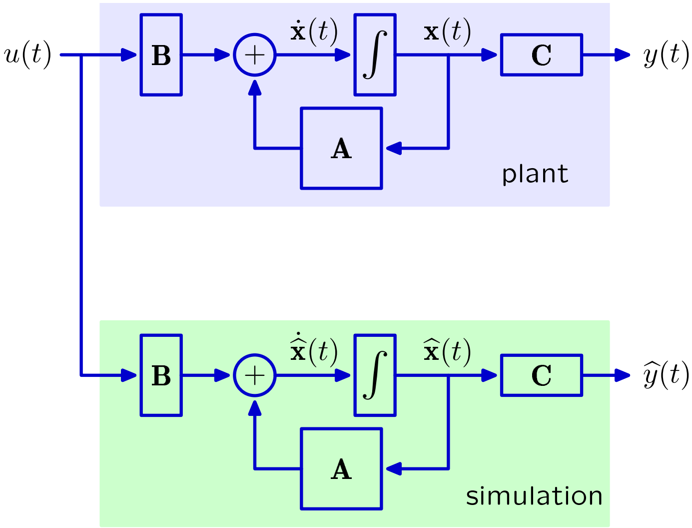
However, to feed back information about $x_1(t)$, we must **measure** $x_1(t)$.

What if it's not possible to measure $x_1(t)$.

Idea: Could we simulate the unmeasured states?

Observers

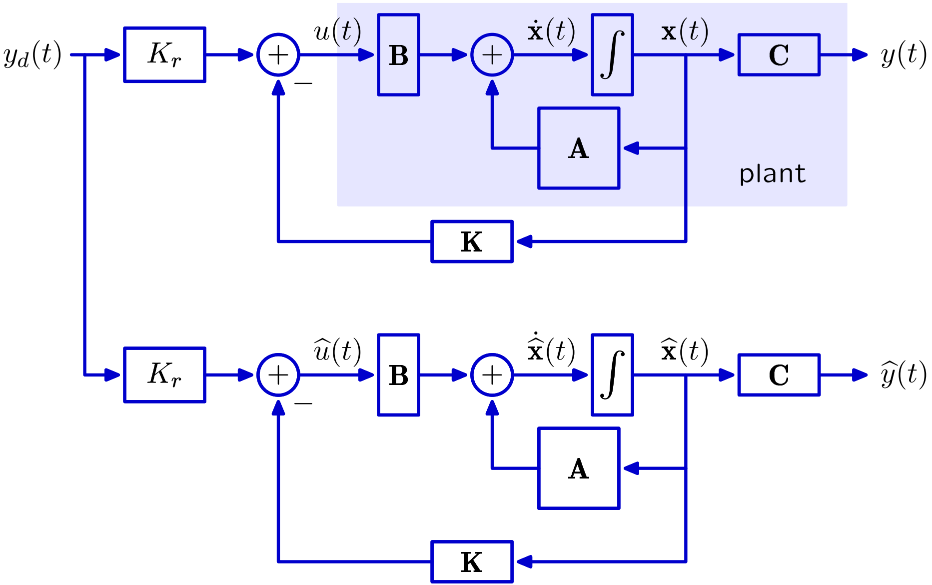
An **observer** is a **simulation** of the plant that is used to provide information about unmeasured states. This **simulation** will be part of the controller!



Observers

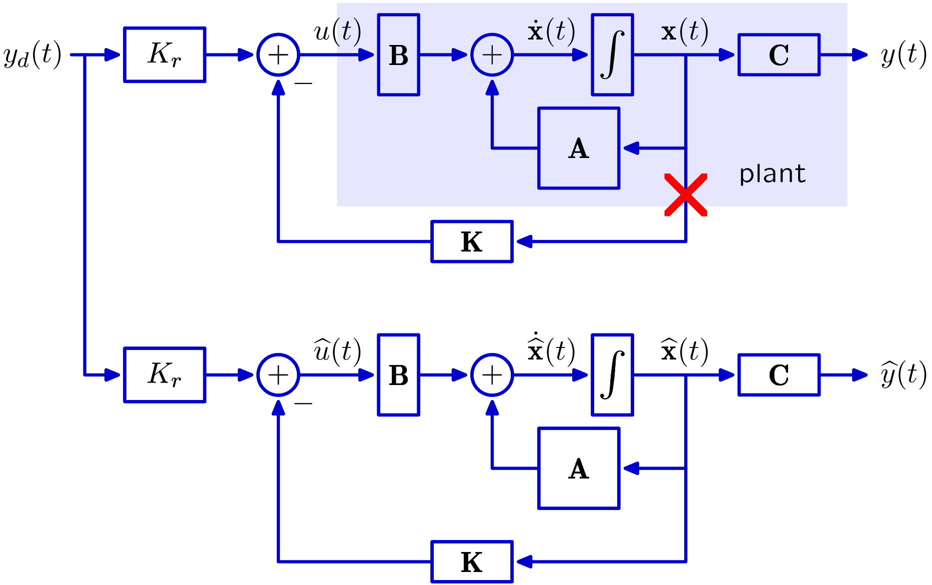
We can build state-space **controllers** for both the plant and the simulation.

If our model of the plant (**A, B, C**) is perfect, then $\widehat{\mathbf{x}}(t) = \mathbf{x}(t)$ and $\widehat{y}(t) = y(t)$.



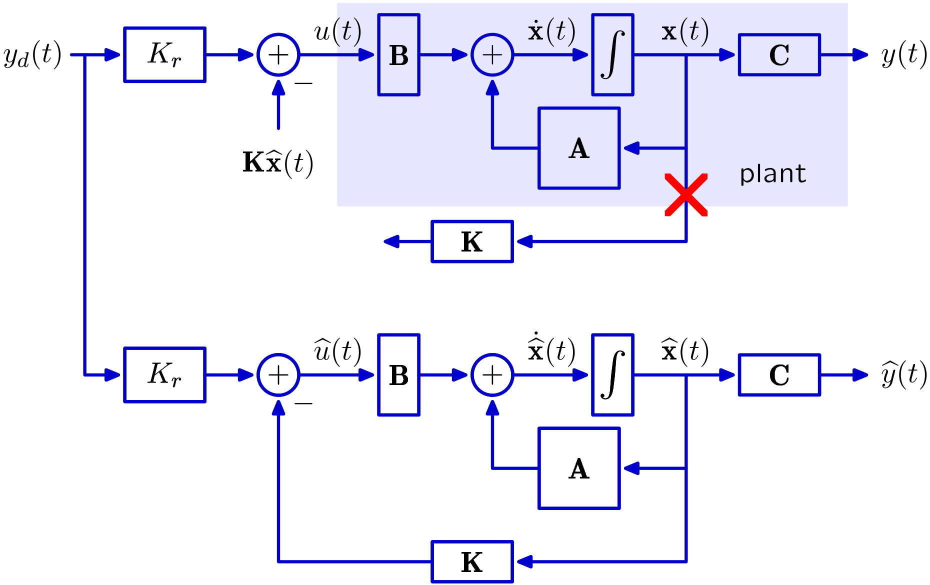
Observers

Recall the problem with designing a state-space controller for the two-springs system: the plant did not provide outputs for all of the states $\mathbf{x}(t)$.



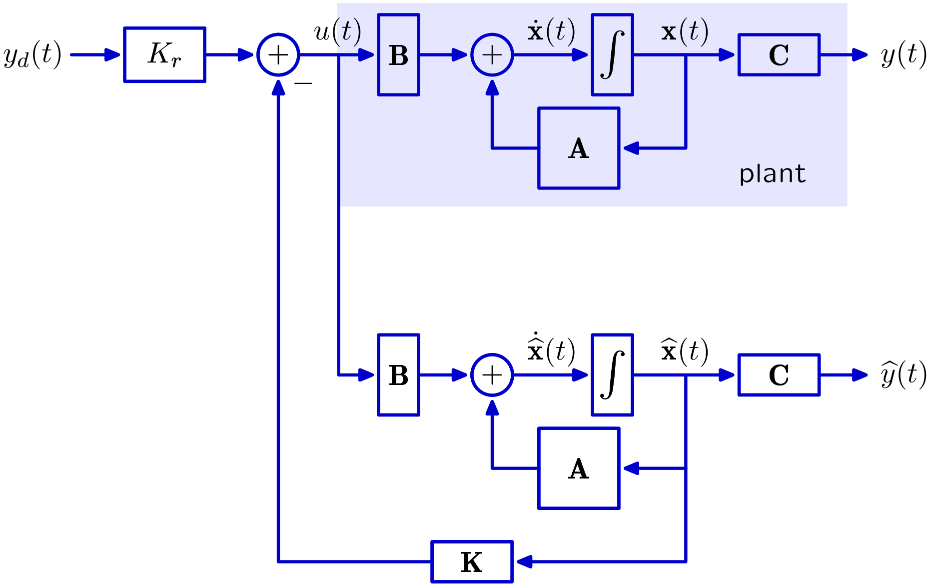
Observers

If our model of the plant (**A**, **B**, **C**) is perfect, then $\widehat{\mathbf{x}}(t) = \mathbf{x}(t)$ and we can replace $\mathbf{K}\mathbf{x}(t)$ with $\mathbf{K}\widehat{\mathbf{x}}(t)$. This substitution also makes $u(t) \equiv \widehat{u}(t)$.



Observers

The resulting structure provides feedback from all **simulated** states $\widehat{\mathbf{x}}(t)$.
But there is a problem. What's the biggest problem with this scheme?

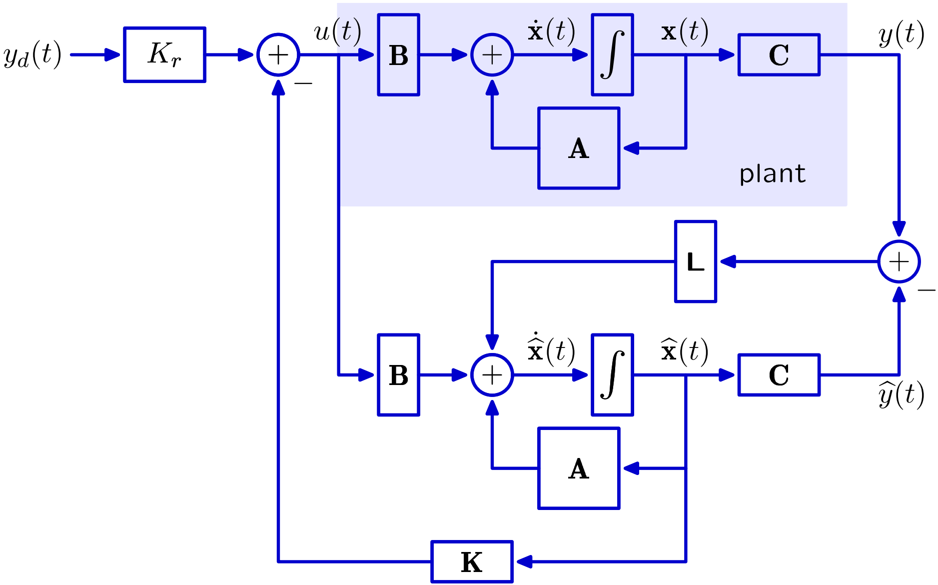


Observers

Fortunately, we can use **feedback** to correct simulation errors!

Calculate the difference between $y(t)$ and $\widehat{y}(t)$.

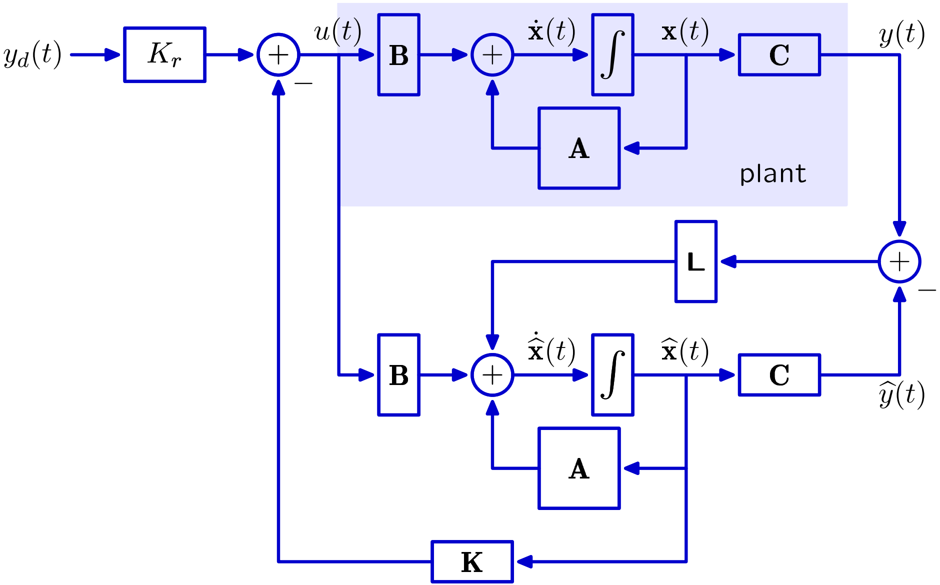
Then use that signal (times $\mathbf{L}$) to correct $\dot{\widehat{\mathbf{x}}}(t)$.



Observers

Plant dynamics: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t)$

Simulation dynamics: $\dot{\widehat{\mathbf{x}}}(t) = \mathbf{A}\widehat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t) + \mathbf{L}(y(t) - \widehat{y}(t))$



Observers

Combined dynamics of the plant and observer.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t)$$

$$\dot{\widehat{\mathbf{x}}}(t) = \mathbf{A}\widehat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\widehat{\mathbf{x}}(t) + \mathbf{B}K_r y_d(t) + \mathbf{L}\Big(y(t) - \widehat{y}(t)\Big)$$

Define $\mathbf{e}(t)$ to be the difference between the plant and simulation states:

$$\mathbf{e}(t) = \mathbf{x}(t) - \widehat{\mathbf{x}}(t)$$

Subtract $\dot{\widehat{\mathbf{x}}}(t)$ from $\dot{\mathbf{x}}(t)$ to find the derivative of $\mathbf{e}(t)$:

$$\dot{\mathbf{e}}(t) = \mathbf{A}\mathbf{e}(t) - \mathbf{L}\Big(y(t) - \widehat{y}(t)\Big) = \mathbf{A}\mathbf{e}(t) - \mathbf{L}\mathbf{C}\mathbf{e}(t)$$

Append the $\dot{\mathbf{x}}(t)$ and $\dot{\mathbf{e}}(t)$ to make a new **combined** state vector:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A}-\mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} K_r y_d(t)$$

Notice that the resulting matrix equation has the same form as the original state evolution equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{x}(t)$ have been extended to include error terms.

Observers

Combined dynamics of the plant and observer.

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} K_r y_d(t)$$

The poles of this system are the roots of its characteristic equation:

$$\left| s\mathbf{I} - \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \right| = 0$$

Because the evolution matrix has block triangular form, the characteristic equation can be factored into two parts:

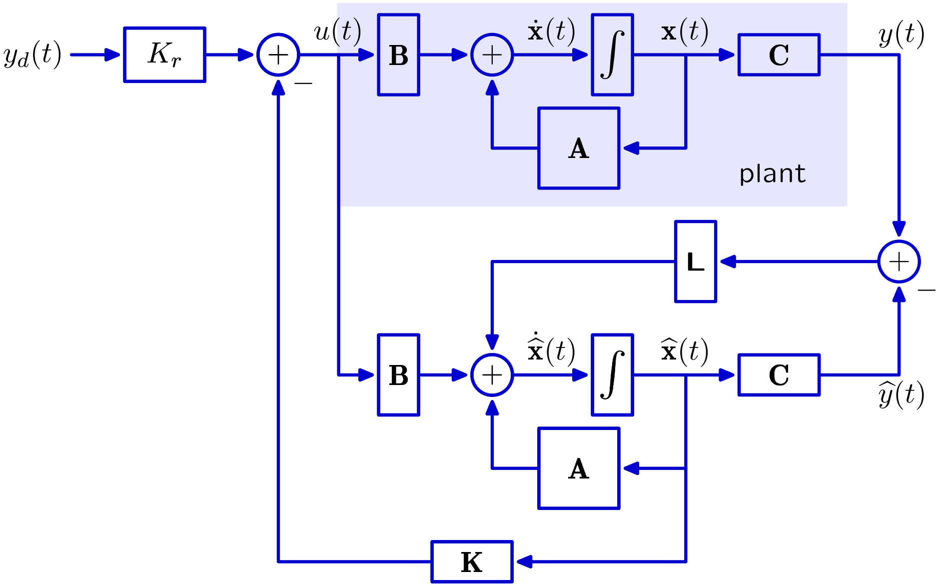
$$\left| s\mathbf{I} - \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \right| = \left| s\mathbf{I} - (\mathbf{A}-\mathbf{BK}) \right| \times \left| s\mathbf{I} - (\mathbf{A}-\mathbf{LC}) \right| = 0$$

and the poles of the augmented system are the union of the poles of the plant and simulation dynamics.

Furthermore, the poles of the plant and observer can be chosen independently, so we can pick an $\mathbf{L}$ to give fast decay of observer state errors (going from $\mathbf{x}(t)$ to $\widehat{\mathbf{x}}(t)$) relative to tracking errors (going from $y_d(t)$ to $y(t)$).

Choosing $\mathbf{L}$

How can we choose $\mathbf{L}$ to make the simulated states $\widehat{\mathbf{x}}(t)$ converge to $\mathbf{x}(t)$?



Choosing L

How can we choose **L** to make the simulated states $\widehat{\mathbf{x}}(t)$ converge to $\mathbf{x}(t)$?

In the normal state-space model, we choose the control vector **K** based on the eigenvalues of plant dynamics:

$$s\mathbf{X}(s) = \mathbf{AX}(s) - \mathbf{BKX}(s) + \mathbf{B}KrY_d$$

Choose **K** to optimize properties of the eigenvalues of **A**-**BK**.

For the observer, we similarly choose the feedback vector **L** based on the eigenvalues of the error dynamics:

$$s\mathbf{E}(s) = \mathbf{AE}(s) - \mathbf{LCE}(s)$$

Choose **L** to optimize properties of the eigenvalues of **A**-**LC**.

The **K** and **L** problems have a similar form – but they are not identical. The form can be made identical by transposition, i.e., optimize the eigenvalues of the transpose $\mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T$ (which are identical to those of **A**-**LC**).

Choosing L

Since optimizing **K** and **L** can be cast into problems with the same form, the optimizations can be solved using the same methods.

```
K = place(A,B,[poles])
L = place(A.',C.',[poles]).'
```

or

```
K = lqr(A,B,Qk,Rk)
L = lqr(A.',C.',Ql,Rl).'
```

Summary

Today we formulated a new approach to control based on **observers**.

- An observer is a simulation of the plant that is part of the controller.
- The biggest challenge in designing an observer is keeping its state up-to-date with that of the plant.
- We can feedback the difference between the measured and simulated outputs to correct the simulated states.