Win, Place, LQR: Post-Lab – Solutions

In this post lab, we will analyze disturbance rejection in state space systems. Recall the general form of state space system is given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where we assume the E matrix is identity. Now suppose there is an unmodelled disturbance that can affect the system state variables $x(t)$. The state space system is modified:

$$\dot{x}(t) = Ax(t) + Bu(t) + Fd(t)$$
$$y(t) = Cx(t) + Du(t)$$

Here $d(t)$ is the disturbance, and F is an $n \times 1$ vector that maps a scalar disturbance to the state variable $x(t)$. Equivalently, the new system has two input variables: $u(t)$ and $d(t)$. Suppose we define a new input vector:

$$u'(t) = \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}$$

Based on this new input vector, we want to write an equivalent state space system:

$$\dot{x}(t) = A'x(t) + B'u'(t)$$
$$y(t) = C'x(t) + D'u'(t)$$

that has the same state variables $x(t)$ and output $y(t)$. Please write the new matrices A', B', C', and D' in terms of the original matrices A, B, C, D, and F. In addition, please write the dimensions of each matrix. Here $x(t)$ has dimension of $n \times 1$.

$A' =$ A : $n \times n$ matrix

$B' =$ $[B\ F]$: $n \times 2$ matrix

$C' =$ C : $1 \times n$ matrix

$D' =$ $[D\ 0]$: 1×2 matrix

Now let's implement a state space controller for designing the input vector $u'(t)$. We let

$$u'(t) = \begin{bmatrix} K_r r \\ d \end{bmatrix} - K'x$$

Here d is the disturbance that we do not have direct control. What is the dimension of the feedback matrix K'?

dimension of K': $2 \times n$ matrix

Let $x(t)$, $y(t)$, and $d_{dist}(t)$ be represented by their transforms:

$$x(t) \to X(s);\; y(t) \to Y(s);\; d(t) \to D_{dist}(s)$$

and solve for the state $X(s)$ of the new controller as a function of the disturbance $D_{dist}(s)$. You can assume $r(t) = 0$. Enter your work in the box below.

$$\dot{x}(t) = A'x(t) + B'u'(t)$$

$$\dot{x}(t) = A'x(t) + B'\left(\begin{bmatrix} K_r r \\ d \end{bmatrix} - K'x\right)$$

$$sX = A'X + B'\left(\begin{bmatrix} KrR \\ D_{dist} \end{bmatrix} - K'X\right)$$

$$(sI - (A' - B'K'))\,X = B'\begin{bmatrix} K_r R \\ D_{dist} \end{bmatrix}$$

$$X = (sI - (A' - B'K'))^{-1} B \begin{bmatrix} KrR \\ D_{dist} \end{bmatrix}$$

$$X = (sI - (A' - B'K'))^{-1} [B\;\; F] \begin{bmatrix} KrR \\ D_{dist} \end{bmatrix}$$

$$X = (sI - (A' - B'K'))^{-1} (BK_r R + FD_{dist})$$

$$X = (sI - (A' - B'K'))^{-1} FD_{dist}$$

Note that in the last step we use $R(s) = 0$.

Next, find the closed-loop transfer function between the output $Y(s)$ and the disturbance $D_{dist}(s)$.

$$y = C'x + D'u = C'(sI - (A' - B'K'))^{-1} FD_{dist}$$

$$H_{close} = C'(sI - (A' - B'K'))^{-1} F$$

Now having the equations, we are going to explore three cases computationally. We will use lab 5 as an example. Specifically, please generate the A, B, C, and D matrices in lab 5. For ease of grading, please use the following parameters:

$$\lambda_e = -125;\; \gamma_{ic} = 1.1;\; \gamma_{ai} = 900;\; \gamma_{ay} = 1000$$

We will assume an external disturbance directly causes a change of the displacement state Δy. So the F matrix is given by:

$$F = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In the first case, we are going to use pole placement just as we did in lab 5. We will completely ignore the matrix F. So you should design your K matrix using the command:

```
K = place(A, B, poles)
```

For ease of grading, please use the pole vector

```
poles = [-190, -200, -20]
```

Please use this `K` vector to simulate the system response to a step disturbance. What are the A, B, C, D, and K matrices?

$$A = \begin{bmatrix} -125 & 0 & 0 \\ 900 & 0 & 1000 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 137.5 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0\ 0\ 1]$$

$$D = 0$$

$$K = [2.0727\ 0.3782\ 9.4545]$$

In the second case, let's design a new K matrix using knowledge of F. Specifically, use the A' and B' matrices you found above to design a new K matrix. Please write the new K matrix.

Hint: the size of this K matrix should be 2×3.

$$K = \begin{bmatrix} 0.5913 & 0.0170 & 0.1312 \\ 12.3693 & 2.8995 & 203.7013 \end{bmatrix}$$

In the third case, please use the integral form of the state space controller with LQR. When formulating the control matrix, please use the following Q and R matrices:

$$Q = \begin{bmatrix} 2000 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = 1$$

What are the A_+, B_+, C_+, D_+, and K_+ matrices?

$$A_+ = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -125 & 0 & 0 \\ 0 & 900 & 0 & 1000 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_+ = \begin{bmatrix} 0 \\ 137.5 \\ 0 \\ 0 \end{bmatrix}$$

$$C_+ = [0\ 0\ 0\ 1]$$

$$D_+ = 0$$

$$K_+ = [44.7214\ 1.1803\ 0.1940\ 6.7805]$$

Please simulate the system response to a step disturbance for all three cases. Please superimpose all three plots on each other. You should see the original controller has the worst disturbance rejection, the controller that considers the F vector performs significantly better, and the integral LQR controller can remove the steady state error.