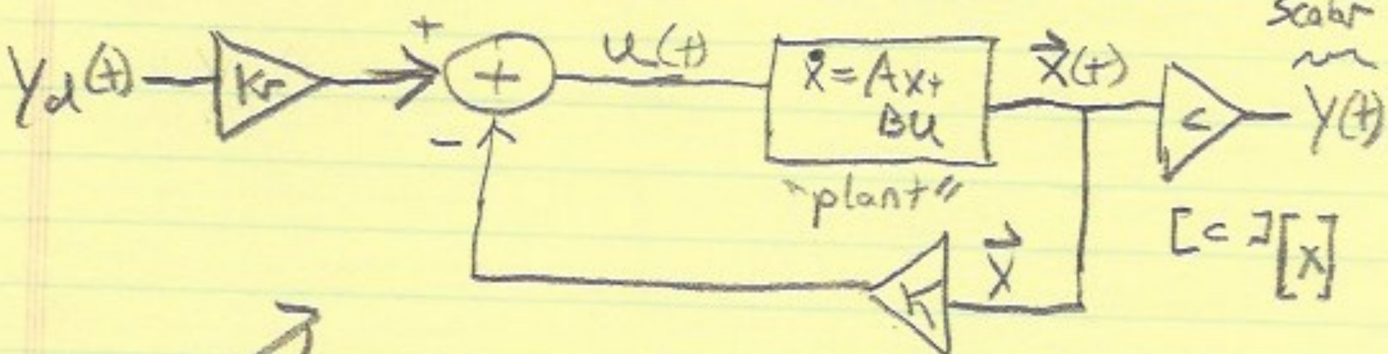
6.310 4/28/25

Today: Observers in C.T. (D.T. after Holiday)

C.T. State Space Reminder



$Kx = [K]\begin{bmatrix} x \end{bmatrix}$
scalar

Diagram

Equation

$$\dot{\vec{x}} = A\vec{x} + Bu \quad u = K_r y_d - K\vec{x}$$

or

$$\dot{\vec{x}} = (A - BK)\vec{x} + BK_r y$$

state

output

AND

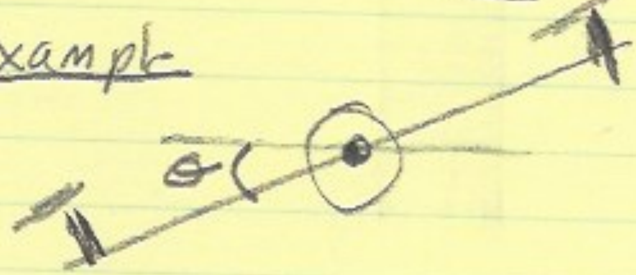
$$y(t) = C\vec{x}(t)$$

Ex y: Arm-angle, umbrella position

2

What if x is $\boxed{\text{not}}$ measurable (or some of x isn't)

Example



propeller arm
-driven

We can measure θ

We can approx $\omega \doteq \dfrac{\theta(t) - \theta(t-\Delta T)}{\Delta T}$

We can $\boxed{\text{not}}$ measure:

$\underbrace{\text{Differential}}$ Propeller speed (proportional to Torque)

Left prop speed − Right prop speed

Set up an Estimator

original system: $\dot{x} = Ax + Bu$

Estimation system (computed using simulation) $\dot{\hat{x}} = A\hat{x} + Bu$

estimate state

We generate u so we know it. same for both systems

Use estimated (e.g. simulated) state for control

$$u(t) = \underbrace{K_r\, y_d(t)}_{\text{known}} - K\hat{x}$$

$\hat{x}$ ← estimated state

"Obvious Problems"

1) is $x(t)\big|_{t=0} = \hat{x}(t)\big|_{t=0}$?
   (Do initial conds match)

2) Must simulate estimation system as part of control

Examine initial condition issue

<u>Define</u> $err(t) = x(t) - \hat{x}(t)$

(1) $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$

(2) $\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t)$

<u>Subtracting</u> (2) from (1) (Assuming A, B model match exactly)

$$\frac{d}{dt}\underbrace{\left(x(t) - \hat{x}(t)\right)}_{err(t)} = A\underbrace{\left(x(t) - \hat{x}(t)\right)}_{err(t)} + \cancel{Bu(t) - Bu(t)}$$

$$\frac{d}{dt}err(t) = A\, err(t)$$

Recall $e^{At} \to 0$ if Re(eigenvals(A)) < 0
↑ nat freqs

So

if $err(t)\big|_{t=0} = x(0) - \hat{x}(0) \neq 0$

and real(eig(A)) < 0 ← strictly state

then $err(t) \to 0$

So initial condition error decays for a strictly stable system!

"Less Obvious Problems"

1) Modeling Errors will cause mismatches
2) Physical system disturbances will not be in estimation (simulation), so No disturbance rejection

Fix with Feedback

We always measure $y(t) = Cx(t)$
if single output system
then $y(t)$ is a scalar
but $x(t)$, $\hat{x}(t)$ are vectors!
How do we use a scalar measurement to correct

Correct $\hat{x}(t)$ using $y(t) = C\vec{x}(t)$

(1) physical system $\dot{x} = Ax + Bu$

(2) Estimation system $\dot{\hat{x}} = A\hat{x} + \underbrace{Bu}_{\text{We KNOW}} + [L](y - \hat{y})$

$y = Cx \qquad \hat{y} = C\hat{x}$

Recall estimation error

$$err(t) \equiv x(t) - \hat{x}(t)$$

(1)-(2)

$$\begin{aligned}
\dot{\vec{e}}rr(t) &= A\vec{e}rr(t) - L(y(t) - \hat{y}(t)) \\
&= A\vec{e}rr(t) - L(Cx(t) - C\hat{x}(t)) \\
&= A\vec{e}rr(t) - LC(\vec{e}rr(t)) \\
&= (A - LC)\vec{e}rr(t)
\end{aligned}$$

Pick L so that $\vec{e}rr(t) \to 0$ fast

Note

$$\begin{array}{c|c} & \overset{n}{\longleftrightarrow} \\ \hline n \updownarrow [B] & [K] \updownarrow 1 \end{array} \qquad \xleftrightarrow[\text{Transpose}]{} \qquad \begin{array}{c|c} & \overset{n}{\longleftrightarrow} \\ \hline n \updownarrow [L] & [C] \end{array}$$

$$\underbrace{(A - BK)}_{} \qquad\qquad (A - LC)$$

$K = place(A, B, poles)$

$K = lqr(A, B, weights)$

$L = place(A^T, C^T, poles)^T$

$= lqr(A^T, C^T, weights)^T$

LQR $L = lqr(A^T, C^T, Q, R)^T$?

what's being minimized?

Pole Placement $L = place(A^T, C^T, poles)^T$

$$eig(A - LC) = eig((A-LC)^T)$$
$$= eig(A^T - C^T L^T)$$

so place works the usual way

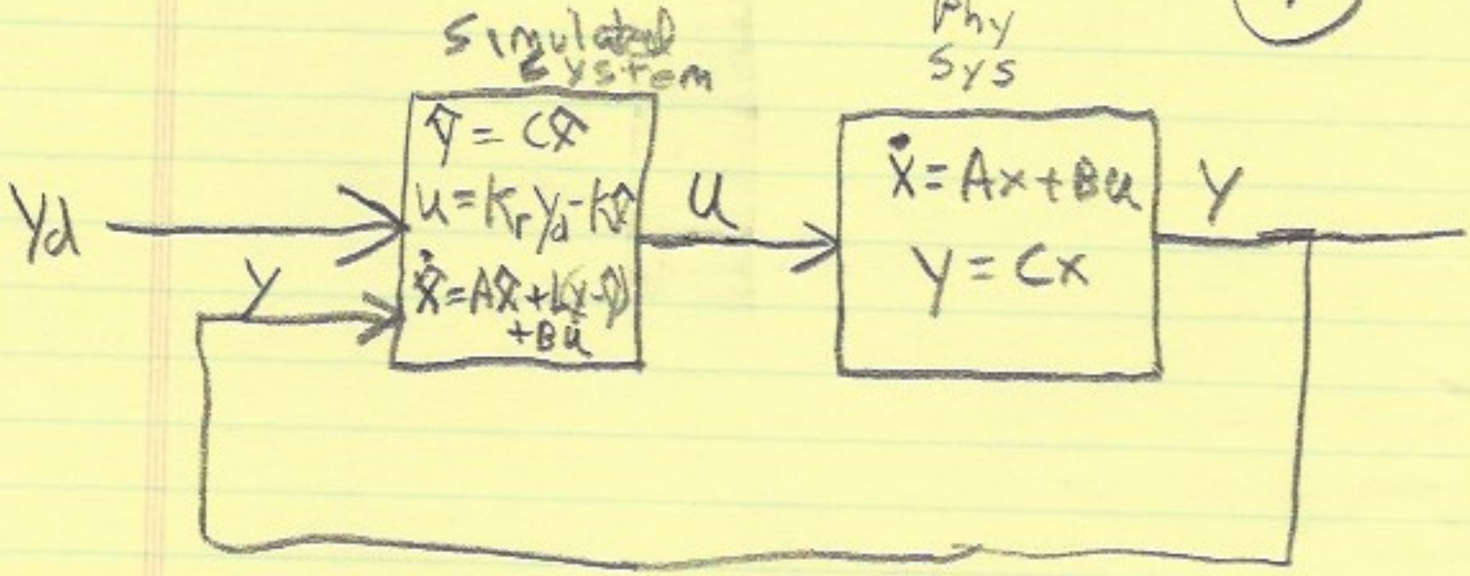
Estimator given L & K

$$u = K_r y_d - K\hat{x}$$

Est $$\dot{\hat{x}} = (A - BK)\hat{x} + K_r y_d + L(y - \overset{C\hat{x}}{\hat{y}})$$

Phys $$\dot{\hat{x}} = (A - BK - LC)\hat{x} + K_r y_d + Ly$$

Hmmm..



<u>Analyzing</u>

(1) Phys $\quad \dot{x} = Ax + Bu = Ax + B(K_r y_d - K\hat{x})$

(2) Est $\quad \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$

$err(t) \equiv x(t) - \hat{x}(t) \qquad \hat{x}(t) = x(t) - err(t)$

(1)-(2) $\quad \dot{err}(t) = (A - LC)\, err(t)$

$$\dot{x} = Ax + BK_r y_d - BK\overbrace{(x(t) - err(t))}^{\hat{x}(t)}$$

$$= (A - BK)x + BK\, err(t) + BK_r y_d$$

$$\begin{bmatrix} \dot{x} \\ \dot{err} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ err \end{bmatrix} + \begin{bmatrix} BK_r \\ 0 \end{bmatrix} y_d$$

Decouples control from estimator!