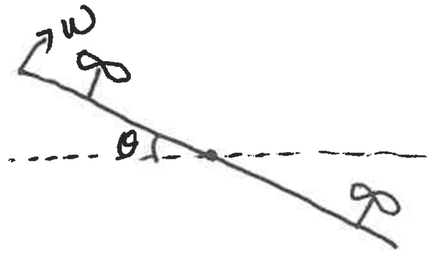
03/12/25.

* Review of Transfer function.

Example: (Note this is NOT exactly the same with our Lab).



A. differential equation (from physics).

$$\frac{d}{dt}\theta(t) = \omega(t). \longrightarrow \Theta = \frac{1}{s}\Omega \quad ①$$

$$\frac{d}{dt}\omega(t) = \beta_w \omega(t) + \gamma c(t). \longrightarrow \Omega = \frac{\gamma}{s-\beta_w} C \quad ②.$$

(β_w comes from friction, air resistance, different from β in the lab).

Guessed solution: $\theta(t) = \Theta e^{st}$, $\omega(t) = \Omega e^{st}$, $\theta_d(t) = \Theta_d e^{st}$

(Why can we do this? — property of linear system,

Recall in last lab, we saw $\sin\omega t$ in, and $\sin\omega t$ out, with only magnitude and phase changed, input/output/command all the same fre

$$c(t) = K_p(\theta_d(t) - \theta(t)) + K_d\frac{d}{dt}(\theta_d(t) - \theta(t)) + K_i\int_0^t(\theta_d(t) - \theta(t))\,dt.$$

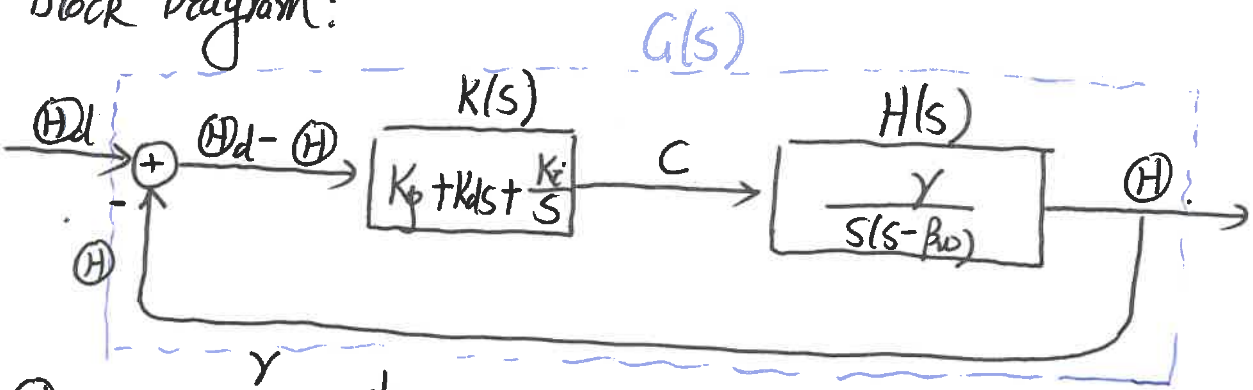
$$\longrightarrow C = K_p(\Theta_d - \Theta) + K_d s(\Theta_d - \Theta) + K_i\frac{1}{s}(\Theta_d - \Theta).$$

B. Transfer function:

① ② ③ $\Rightarrow$ $\Theta = \overbrace{\frac{\gamma}{s(s-\beta\omega)}}^{H(s)} \overbrace{\left(K_p + K_d s + \frac{K_i}{s}\right)}^{K(s)} (\Theta_d - \Theta)$.

Simplify $\boxed{\Theta = \frac{H(s)\,K(s)}{1+H(s)\,K(s)}\,\Theta_d.}$ Black's formula.

C. Block Diagram:



$\Theta = \frac{\gamma}{s(s-\beta\omega)}\,C$, $\quad C = \left(K_p + K_d s + \frac{K_i}{s}\right)(\Theta_d - \Theta)$

Closed loop gain: $G(s) = \frac{\Theta(s)}{\Theta_d(s)} = \frac{H(s)\,K(s)}{1 + H(s)\,K(s)}$ → Transfer function of feedforward loop

*. Use Transfer function to understand time domain behavior

For simplicity, use the example of "P" control, $K_p \neq 0$, $K_d = K_i = 0$.

$$\Theta(s) = \frac{K_p \gamma}{s^2 - \beta\omega s + K_p \gamma}\,\Theta_d(s).$$

Two cases: ① $\cancel{\Theta_d}$ $\theta_d(t) = \Theta_d e^{st} = 0$, no input.

$\Theta(s)$ can still be non-zero if denominator $= 0$.

characteristic equation: $s^2 - \beta\omega s + K_p \gamma = 0$.

Roots: $s_{1,2} = \dfrac{\beta\omega \pm \sqrt{\beta\omega^2 - 4K_p\gamma}}{2}$ natural frequency.

This means $\theta(t)$ can have components $e^{s_1 t}$ or $e^{s_2 t}$ under zero input.

Generally $\theta(t) = \Theta_1 e^{s_1 t} + \Theta_2 e^{s_2 t}$ — zero input response. natural res

② Denominator $\neq 0$,

$$\theta_d = \Theta_d e^{st} \longrightarrow \boxed{G(s)} \longrightarrow \theta(t) = \Theta e^{st} = G(s)\Theta_d e^{st}.$$

Compared with input $\theta_d(t)$, output $\theta(t)$ is scaled by $G(s)$, but keeps the same frequency (complex frequency), or the same e^{st} form.

* Example of Sinusoidal Steady State.

Input: $\theta_d(t) = \Theta_d \cos\omega t$

Output: $\theta(t) = \Theta$?

Euler's formula: $\theta_d(t) = \frac{1}{2}\Theta_d \left(e^{j\omega t} + e^{-j\omega t}\right)$.

$$\tfrac{1}{2}\Theta_d e^{j\omega t} \longrightarrow \boxed{G(s)} \longrightarrow \tfrac{1}{2}\Theta_d G(s)\Big|_{s=j\omega} e^{j\omega t}$$

$$\tfrac{1}{2}\Theta_d e^{-j\omega t} \longrightarrow \boxed{G(s)} \longrightarrow \tfrac{1}{2}\Theta_d G(s)\Big|_{s=-j\omega} e^{-j\omega t}$$

Superposition: $\theta(t) = \frac{1}{2}\Theta_d \left(G(j\omega)e^{j\omega t} + G(-j\omega)e^{-j\omega t}\right)$. (note compl conjugate

$= \mathrm{Re}\{\Theta_d\, G(j\omega)\, e^{j\omega t}\}$. $\quad (G(j\omega) = |G(j\omega)|\, e^{j\angle G(j\omega)})$.

$= \Theta_d\, |G(j\omega)|\, \cancel{e} \cos(\omega t + \angle G(j\omega))$.

$\Rightarrow$ Compared with $\theta_d(t) = \Theta_d \cos\omega t$, output $\theta(t)$'s magnitude change by $|G(j\omega)|$, its phase changes by $\angle G(j\omega)$.

See handout for example plot of $\left| G(s)|_{s=j\omega} \right|$ and $\angle G(s)|_{s=j\omega}$.

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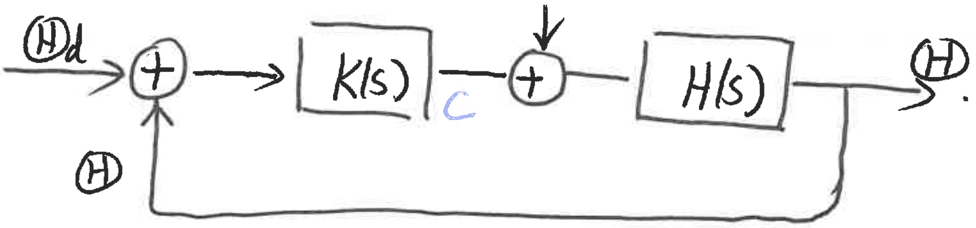
* Disturbance Rejection.

W ← disturbance, like another force/command



$\Theta = H(s)\,(W + C)$.

$C = K(s)\,(\Theta_d - \Theta)$.

$$\Rightarrow \Theta = \underbrace{\frac{K(s)H(s)}{1+K(s)H(s)}}_{G(s)}\Theta_d + \underbrace{\frac{H(s)}{1+K(s)H(s)}}_{G_d(s)} W.$$

$G(s)$ and $G_d(s)$ are transfer function for input and disturbance, separate

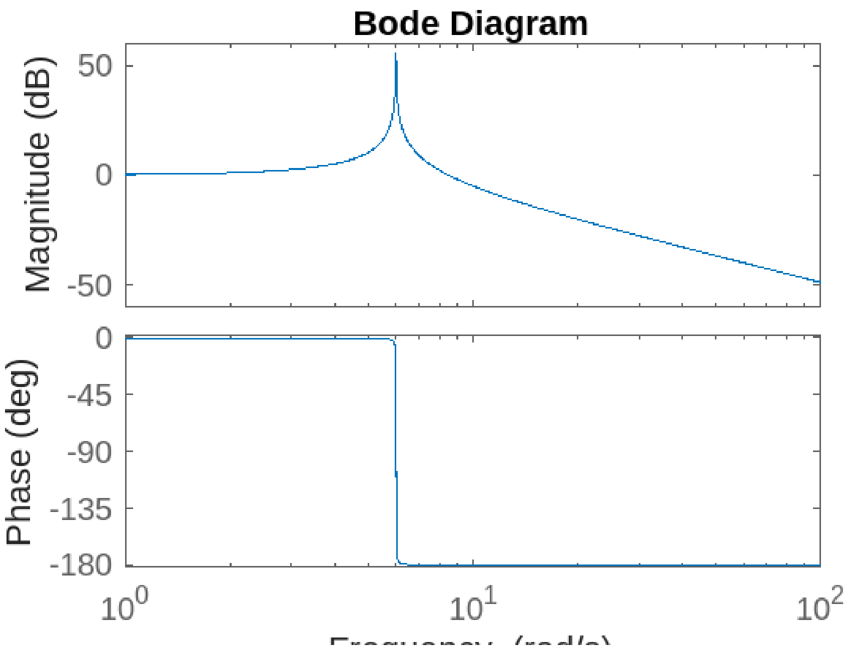
What's the common feature and different feature for $G(s)$ and $G_d(s)$?

Does this explain why you see the same peak frequency between sinusoidal drive and tapping drive last week?

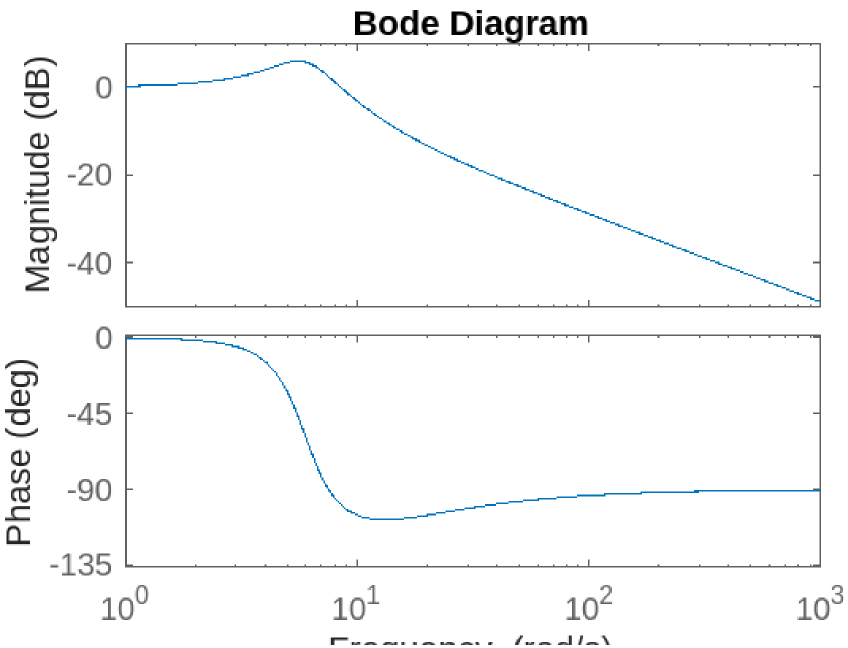
$$G(s) = \frac{(K_p + K_D s + K_I)\frac{\gamma}{s(s-\beta_\omega)}}{1 + (K_p + K_D s + K_I)\frac{\gamma}{s(s-\beta_\omega)}}$$

γ=36; β=-0.01

Kp = 1;
Kd = Ki = 0;



Kp = 1;
Kd = 0.1; Ki = 0;



Kp = 1;
Kd = 0.1; Ki = 2;