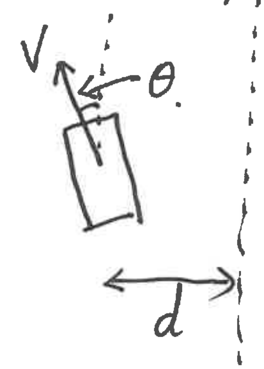
04/09/25.

* State space model vs. transfer function.
* Natural frequency and characteristic equation.

1. State Feedback:

Line follower Example:

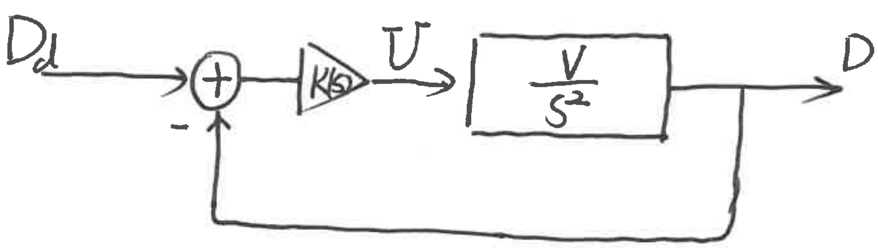


$$\frac{d}{dt} d(t) = V\theta(t). \qquad \Longrightarrow \qquad D = \frac{V}{s}\Theta$$

$$\frac{d}{dt}\theta(t) = u(t) \qquad \Longrightarrow \qquad \Theta = \frac{1}{s}U.$$

↑ rotation rate, torque, cmd.

Previously we introduced transfer function and Block diagram.



So far we designed two types of controllers $K(s)$:

① PID. $K(s) = K_p + K_d s + \frac{K_i}{s}$

② Lead controller: $K(s) = K_o \frac{S_p}{S_z} \cdot \frac{s - S_z}{s - S_p}$.

We also learned techniques to implement these designs:

a). $G(s) = \dfrac{K(s)H(s)}{1+K(s)H(s)}$ → poles/natural frequencies to be on left half plane

b). $K(j\omega)H(j\omega)$: phase margin to be positive.

Questions: A. How many turning knobs are most appropriate?

$$K_{PD} = K_P + K_D s + (K_{D_2} s^2 + K_{D_3} s^3 + \cdots) \quad \text{why not?}$$

B. How to get a guaranteed "optimal" controller?

So far trial and error at best.

Let's step back to differential equation, write in matrix form, (like in DT case).

$$\underbrace{\frac{d}{dt}\begin{pmatrix} d(t) \\ \theta(t) \end{pmatrix}}_{\text{1st order derivative of state variable.}} = \underbrace{\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}}_{\text{A matrix (open loop) system}} \underbrace{\begin{pmatrix} d(t) \\ \theta(t) \end{pmatrix}}_{\text{state variable}} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{B matrix (open loop) input matrix}} u(t).$$

Or more generally $\overset{1}{\overleftrightarrow{\ }}$ $N \updownarrow x(t) = \begin{pmatrix} d(t) \\ \theta(t) \end{pmatrix}$ $N \times 1$ matrix / vector.

$$\frac{d}{dt} x(t) = \underset{N\updownarrow}{\overset{\overleftrightarrow{N}}{A}}\ \underset{N\updownarrow}{\overset{\overleftrightarrow{1}}{x(t)}} + \underset{N\updownarrow}{\overset{\overleftrightarrow{1}}{B}}\, u(t) \leftarrow \text{scalar.}$$

Output: $y(t) = d(t) = \underbrace{(1 \quad 0)}_{\text{C matrix, output matrix}} \begin{pmatrix} d(t) \\ \theta(t) \end{pmatrix}$

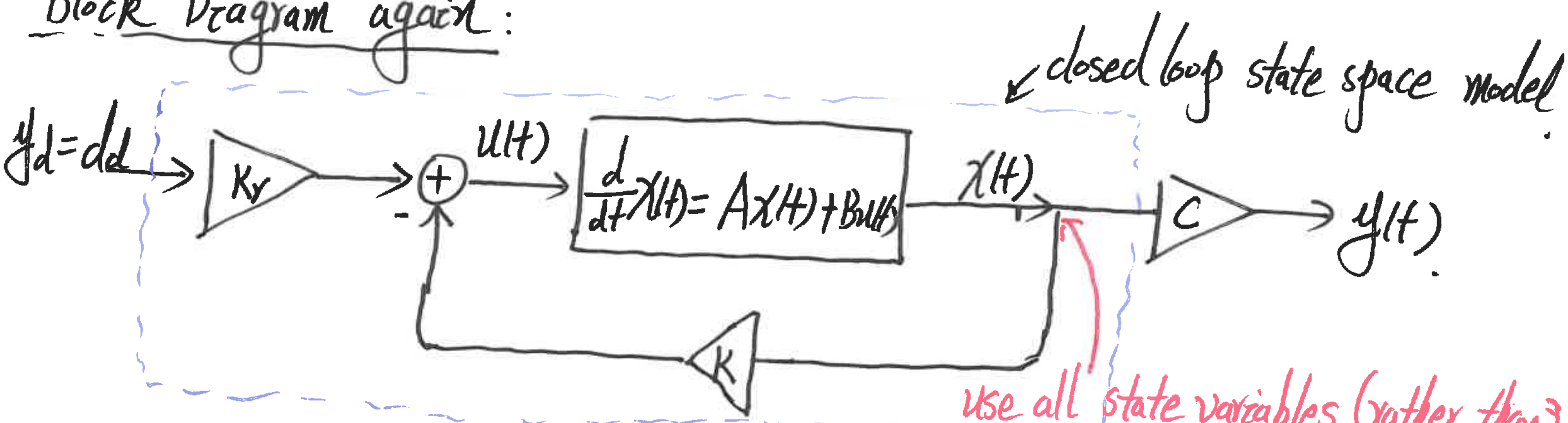
More generally $\underset{\text{scalar}}{y(t)} = \underset{1 \times N}{C}\, \underset{N \times 1}{x(t)}$

Controller: $u(t) = \underset{\uparrow}{K_r} d_d - K_1 d(t) - K_2 \theta(t) = K_r \underset{\text{input}}{d_d} - \underbrace{(K_1 \quad K_2)}_{\text{K matrix}} \begin{pmatrix} d(t) \\ \theta(t) \end{pmatrix}$.

an extra coefficient to get good steady state value, discuss later

More generally $u(t) = K_r d_d - \underset{1 \times N}{K}\, \underset{N \times 1}{x(t)}$

Block Diagram again:



use all state variables (rather than only final output) for feedback.

Getting the closed-loop state space equation by substituting $u(t)$:

$$\frac{d}{dt}x(t) = Ax(t) + B\left(K_r y_d - K x(t)\right)$$

$$\frac{d}{dt}x(t) = \underbrace{(A - BK)}_{A_c\text{: closed loop system matrix}} x(t) + \underbrace{B K_r}_{B_c\text{: closed loop input matrix}} y_d .$$

Sometimes, to generalize, add additional matrix E on L.H.S.

$$E \frac{d}{dt} x(t) = A_c x(t) + B_c y(t),$$ usually $E = I$ = identity matrix.

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* Natural frequency.

Let's solve the state space equation, consider natural response only:

Guessed solution $$x(t) = \begin{pmatrix} d(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} D \\ \Theta \end{pmatrix} e^{st} = X e^{st}$$

$$sX = (A - BK) X \Rightarrow s \text{ is the eigen value of matrix } A_c = A - BK.$$

Line follower example: $$A_c = A - BK = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} (K_1 \quad K_2)$$

$$= \begin{pmatrix} 0 & V \\ -K_1 & -K_2 \end{pmatrix}.$$

$\text{eig}(A_c) = ?$ $$\det(sI - A_c) = \begin{vmatrix} s & -V \\ K_1 & s+K_2 \end{vmatrix} = 0.$$

$s^2 + K_2 s + K_1 V = 0$: characteristic equation, $s_{1,2} =$, natural freq.

Recall results from transfer function: (PD).

$$G(s) = \frac{(K_P + K_D s)\frac{1}{s^2}}{1 + (K_P + K_D s)\frac{1}{s^2}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P}$$ → set to zero for nat. freq.

We see the characteristic equation and natural freq. from the two approaches.

are the same.

Additionally, $K_p \longleftrightarrow K_1 V$: gain on state variable $d(t)$

$K_d \longleftrightarrow K_2$: gain . . . . . . $\theta(t)$.

not surprising as $\theta(t) = \frac{d}{dt} d(t)$.

But state feedback control allows us to generalize. think about system with additional degree of freedom: $d(t)$, $\dot{d}(t)$, $\ddot{d}(t)$, etc. and robot with many degrees of freedom.