04/02/25.

* Asymptotic Behavior in Bode Diagram

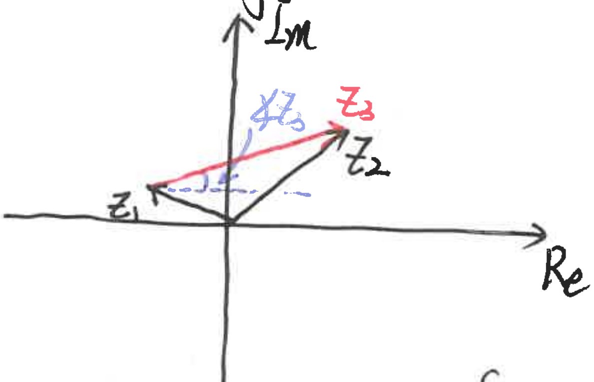
*. Lead Compensation.

1. Review of Complex functions.

Some basic properties: $z_{1,2}$ are complex numbers.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \measuredangle\left(\frac{z_1}{z_2}\right) = \measuredangle z_1 - \measuredangle z_2.$$

Geometric representation:



$z_3 = z_2 - z_1$

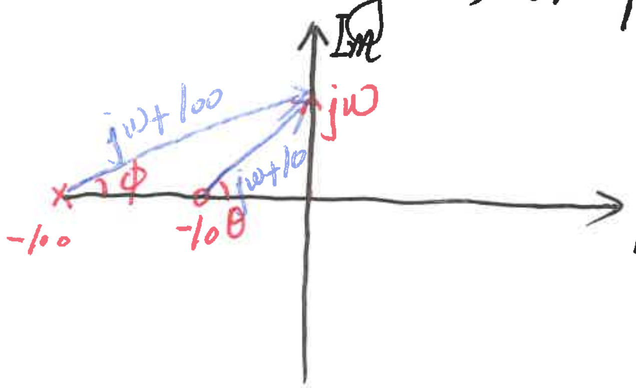
$|z_3|$: length of the line segment

$\measuredangle z_3$: angle with respect to $+x$.

Example: $L(j\omega) = \frac{s_p}{s_z} \cdot \frac{j\omega - s_z}{j\omega - s_p}$ with $s_p = -100$, $s_z = -10$.

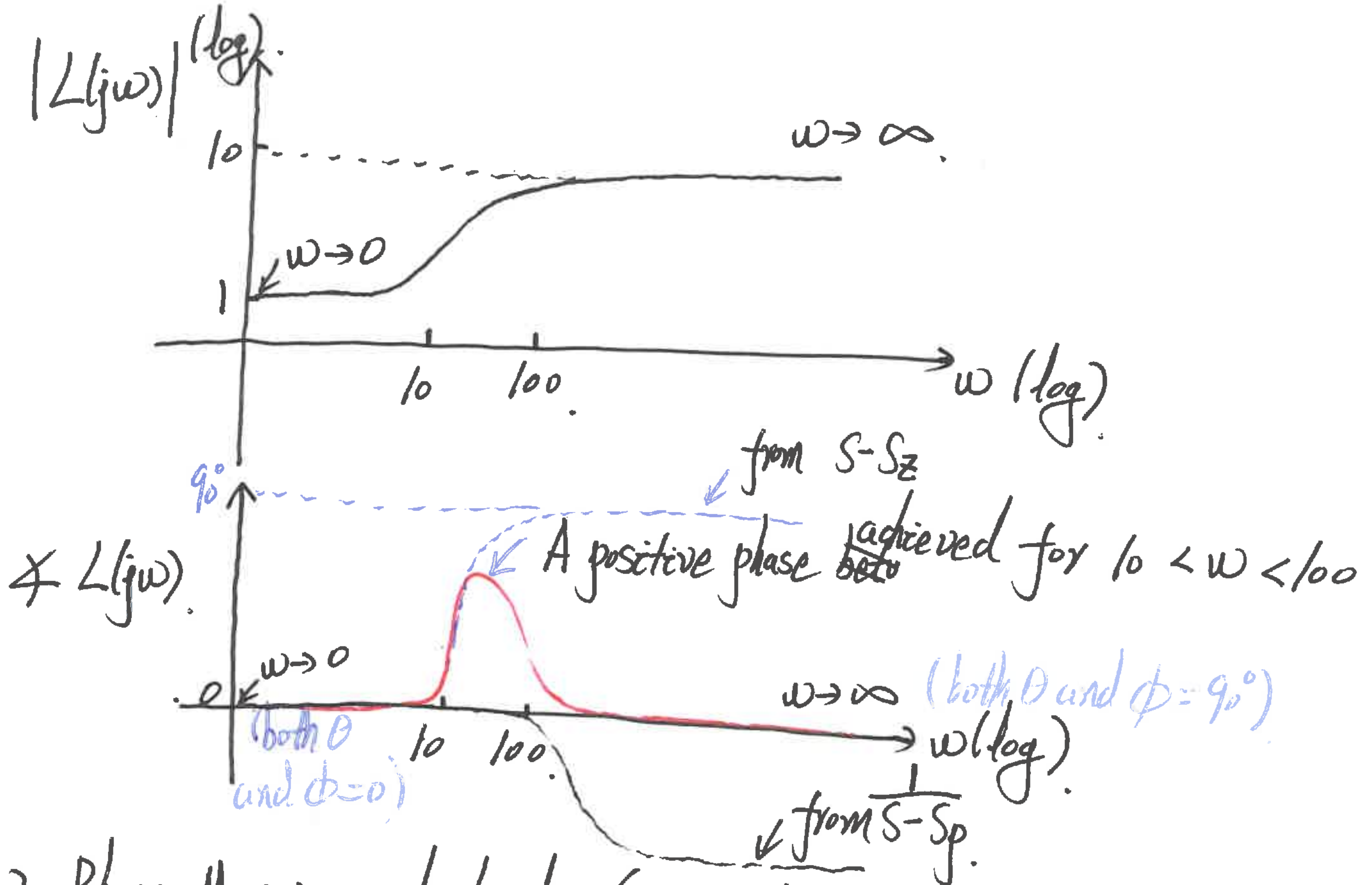
i.e. $L(j\omega) = 10 \cdot \frac{j\omega + 10}{j\omega + 100}$.

Draw Bode diagram, or $|L(j\omega)|$ and $\measuredangle L(j\omega)$ as ω goes from 0 to ∞.



$|L(j\omega)|$: the ratio of the two segments.

$\measuredangle L(j\omega)$: $\theta - \phi$.



2. Phase Margin and Lead Compensation

$$G(s) = \frac{K(s)H(s)}{1 + K(s)H(s)}$$

Considering sinusoidal steady state response, $x_d(t) = X_d e^{j\omega t}$.

Frequency response: $G(j\omega) = \dfrac{K(j\omega)H(j\omega)}{1 + K(j\omega)H(j\omega)}$

Previously learned:

A. Want $|K(j\omega)H(j\omega)|$ to be large, for good tracking and rejecting disturb.

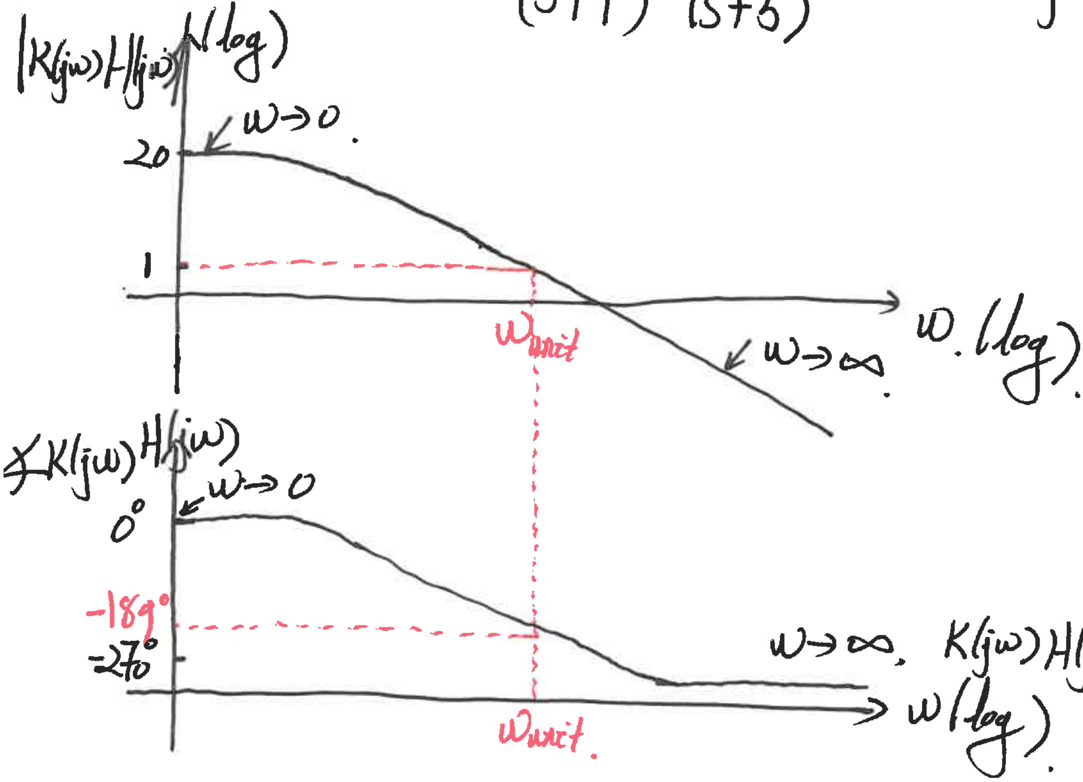
B. Want to avoid $K(j\omega)H(j\omega) = -1$ $\Longrightarrow$ unstable, large sensitivity to disturb.

Equivalently, want when $|K(j\omega)H(j\omega)| = 1$, $\angle K(j\omega)H(j\omega) \neq -180°$.

In real world, usually want $\angle K(j\omega)H(j\omega)$ to be less negative than $-180°$.

$$\text{Phase Margin} = \angle K(j\omega)H(j\omega) - (-180°)\Big|_{\omega_{\text{unity-gain}}}$$

Example: $H(s) = \dfrac{1}{(s+1)^2(s+5)}$ $\quad K_p = 100$.



$\omega \to \infty$, $K(j\omega)H(j\omega) \sim \dfrac{100}{(j\omega)^3} \to \angle = -270°$.

Phase Margin $= -189° - (-180°) = -9° \longrightarrow$ Bad! unstable

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Improved controller: A) PD controller.

$$K_{PD}(s) = K_p + K_D s = 100(s + 0.1)$$

See plots in handout. Phase margin $= 39.8°$. $\leftarrow$ good, stable step response.

This is because K_{PD} boost phase by $\sim 90°$

Downside: $|K(j\omega)| \to \infty$ as $\omega \to \infty$, amplify noise.

B). Lead Compensation:

$$K_L(s) = K_p \frac{S_p}{S_z} \frac{S - S_z}{S - S_p}$$

$\omega \to 0$, $K_L \sim K_p$, large gain for tracking

$\omega \to \infty$, $K_L \sim K_p \frac{S_p}{S_z}$, finite gain, not amplify noise too much!

At intermediate ω (where needed). K_L provides a phase boost (see plots).

Question: How to pick S_p and S_z values to optimize phase margin?

Some considerations: ① $|S_z| < |S_p|$ or $|S_z| > |S_p|$?

② $|S_z|$ and $|S_p|$ separated far apart or close to each other?

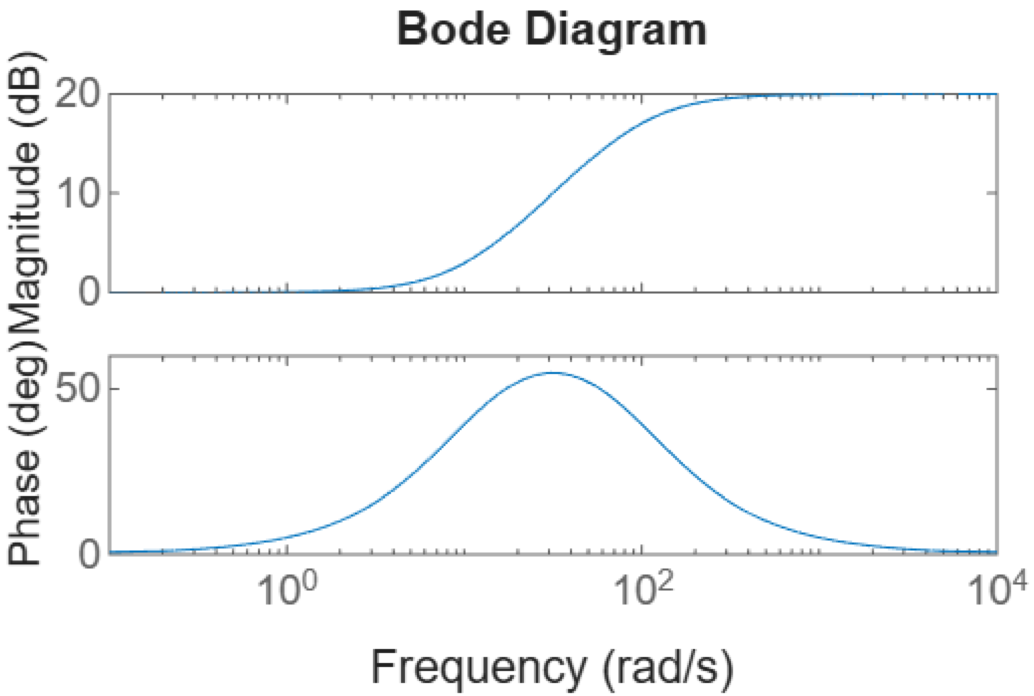
③. $|S_z|$, $|S_p|$ and ω_{unit} relationship?

See examples in plots!

Asymptotic behavior in Bode diagram

$$L(s) = \frac{s_p}{s_z}\frac{s - s_z}{s - s_p}$$

```
s = tf('s'); sz = -10; sp = -100; L = sp*(s-sz)/(s-sp)/sz;
figure; bode(L);
```



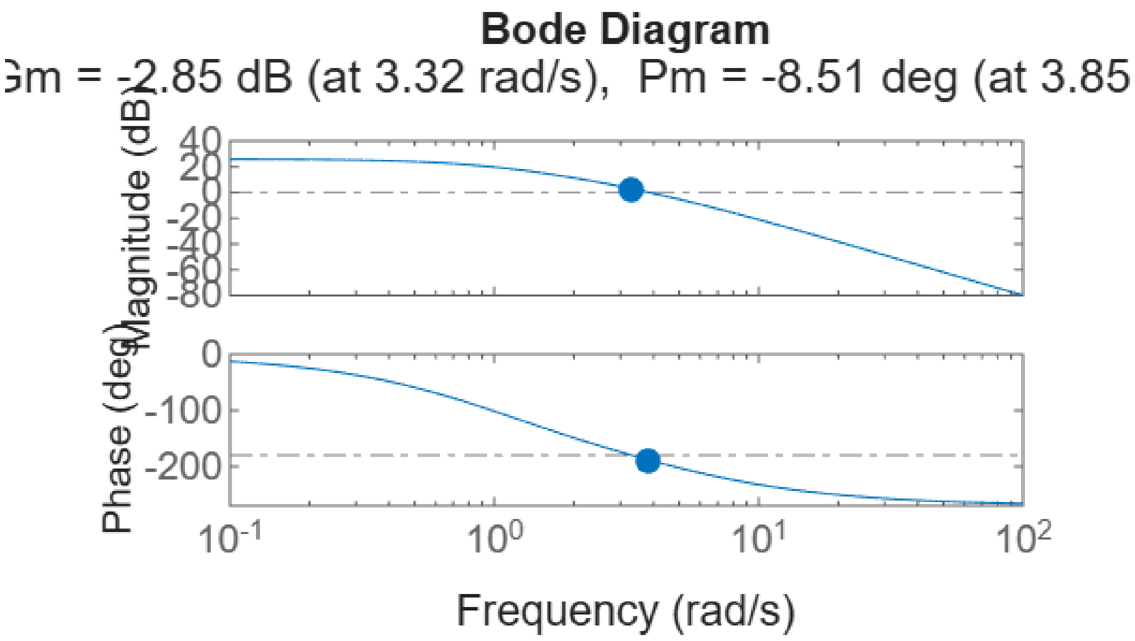
Example

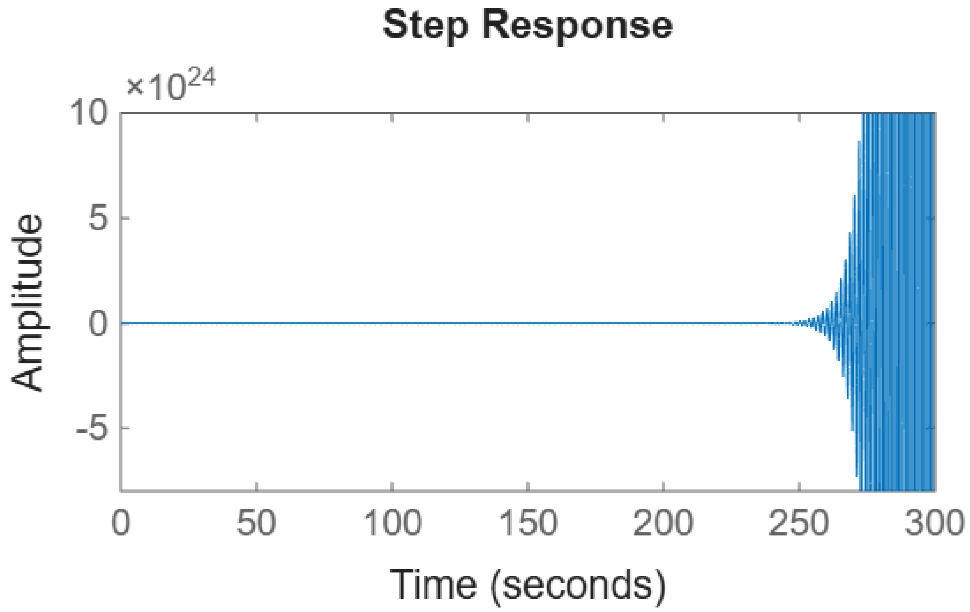
$$H(s) = \frac{1}{(s+1)^2(s+5)}$$

P controller

$$K_p = 100$$

```
H = 1/(s+1)^2/(s+5); K_P = 100; margin(K_P*H); step(feedback(K_P*H, 1))
```

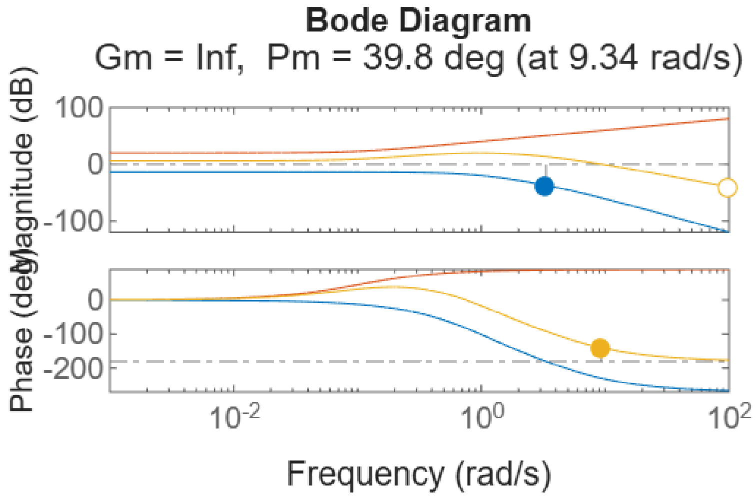


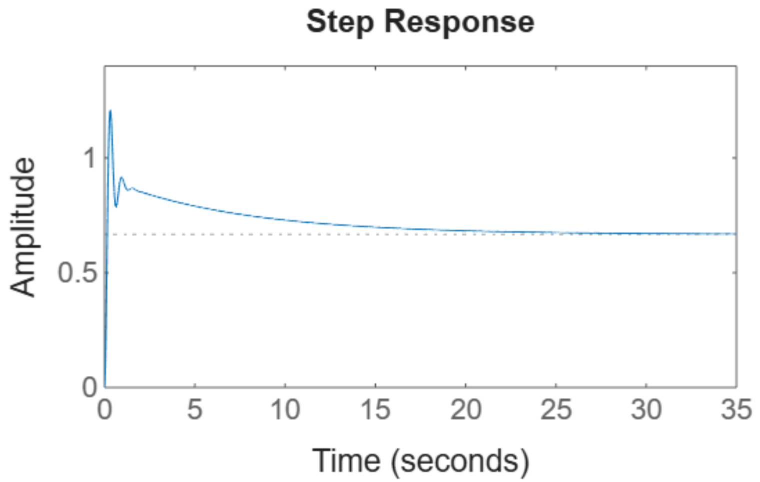


PD controller

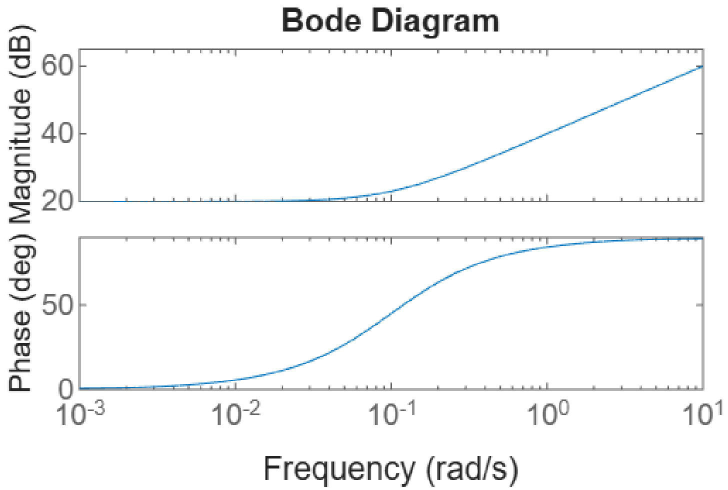
$$K_{PD}(s) = K_p + K_D s = 100(s + 0.1)$$

```
K_PD = 100*(s+0.1);figure; margin(K_PD*H); step(feedback(K_PD*H, 1))
```





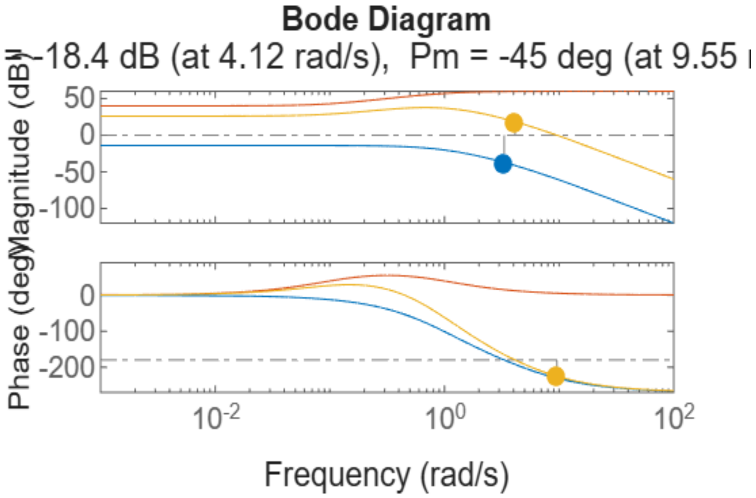

```
bode(K_PD)
```

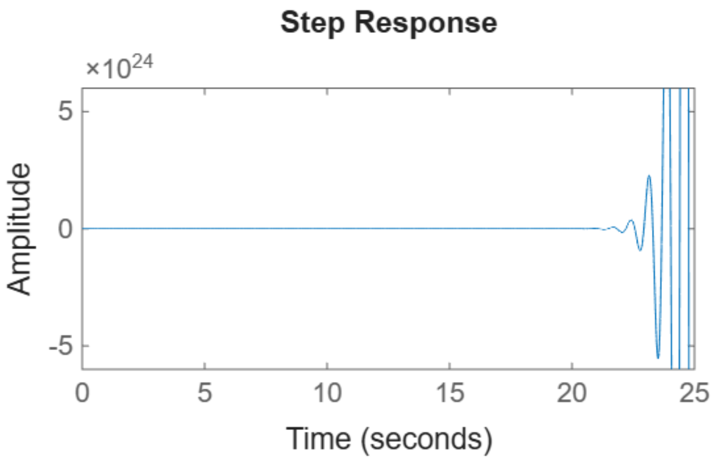


Lead compensation

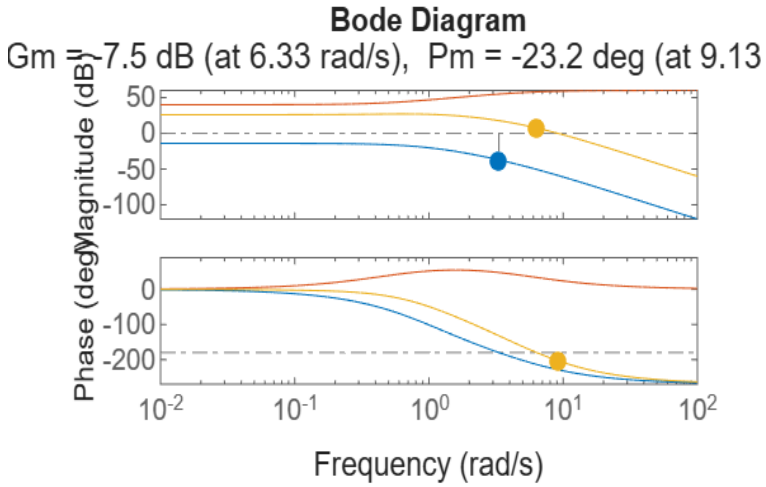
$$K_L(s) = K_p \frac{s_p}{s_z} \times \frac{s - s_z}{s - s_p}$$

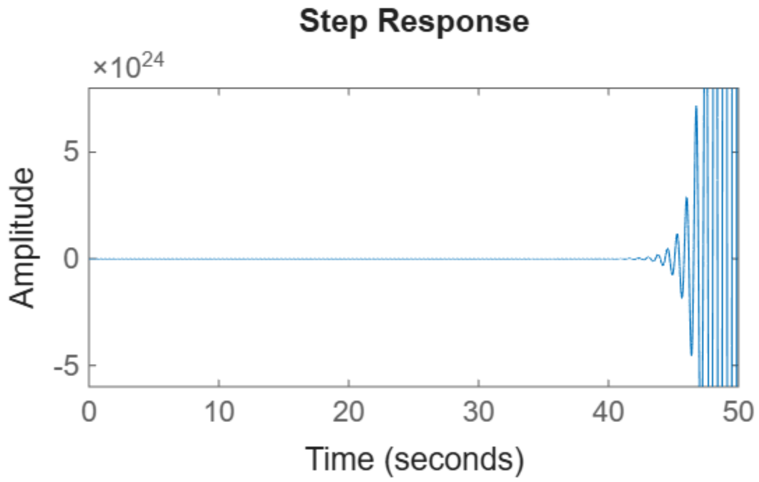
```
sz = -0.1; sp = -1; K L = K P*sp*(s-sz)/(s-sp)/sz; margin(K_L*H); step(feedback(K_L*H,1));
```



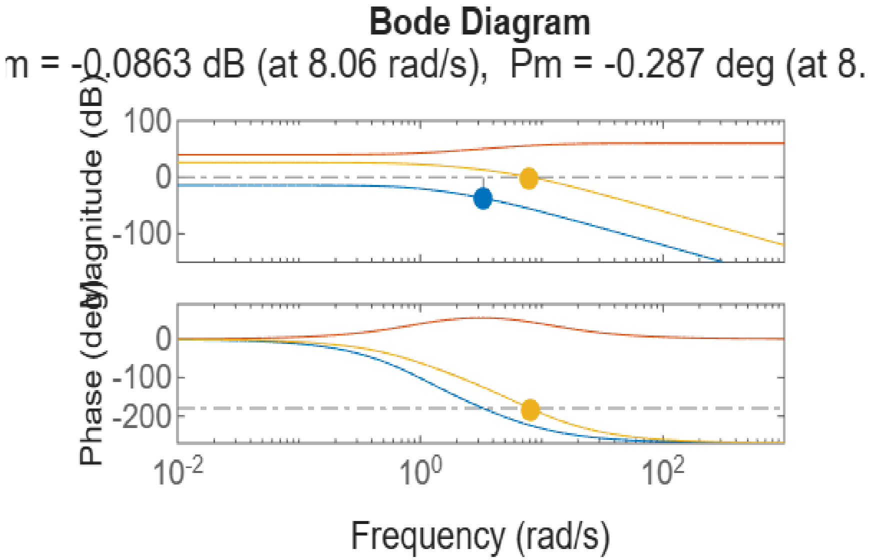


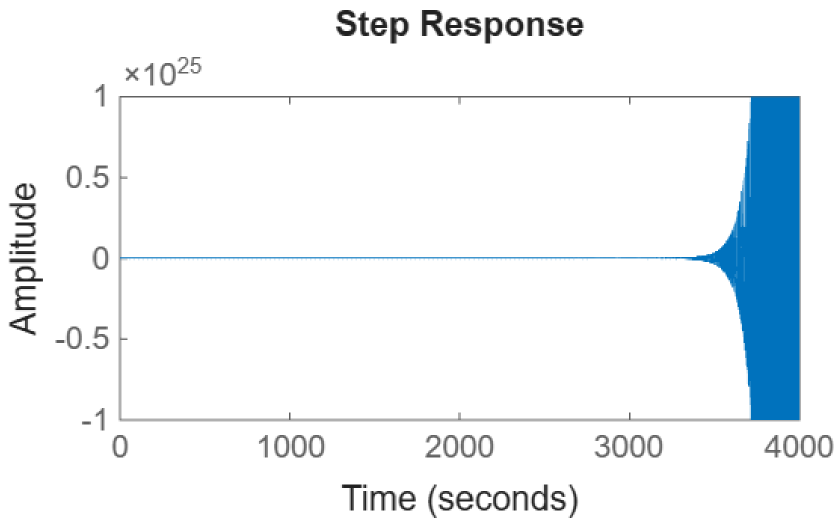

```
sz = -0.5; sp = -5; K_L = K_P*sp*(s-sz)/(s-sp)/sz; margin(K_L*H); step(feedback(K_L*H,1));
```



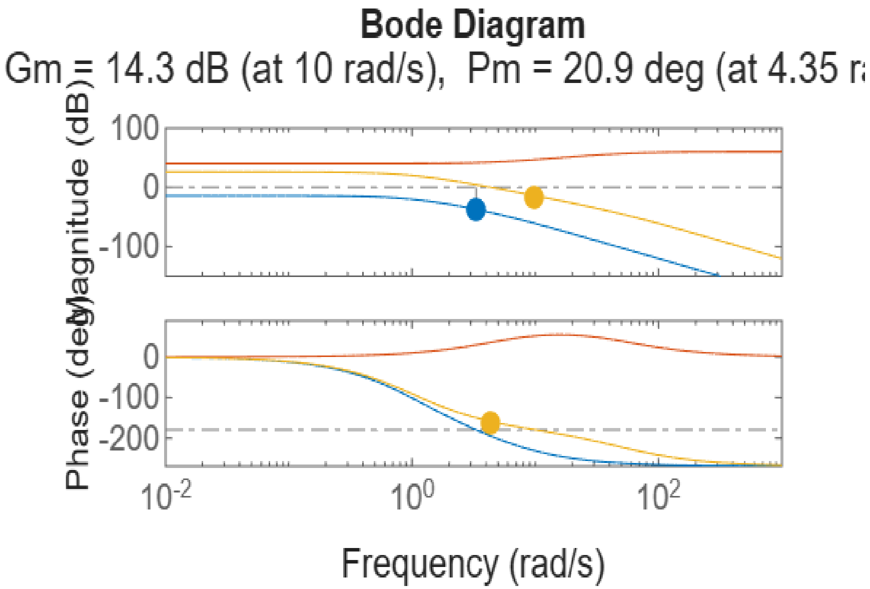


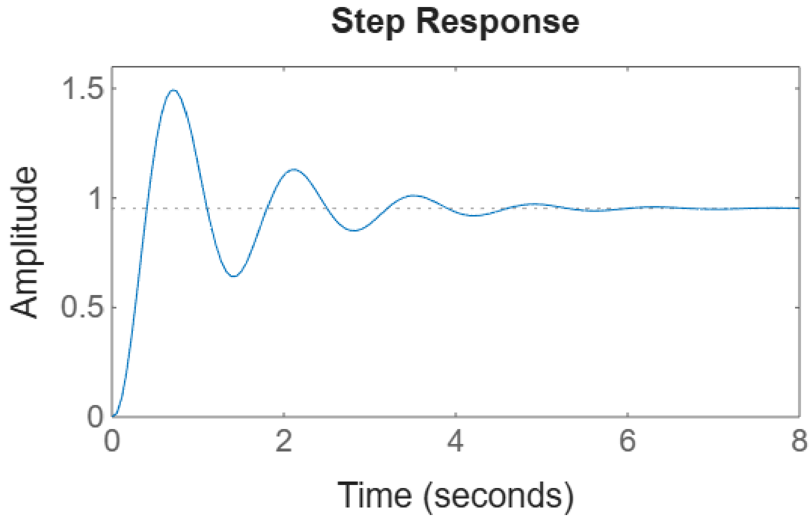
```
sz = -1; sp = -10; K_L = K_P*sp*(s-sz)/(s-sp)/sz; margin(K_L*H); step(feedback(K_L*H,1));
```



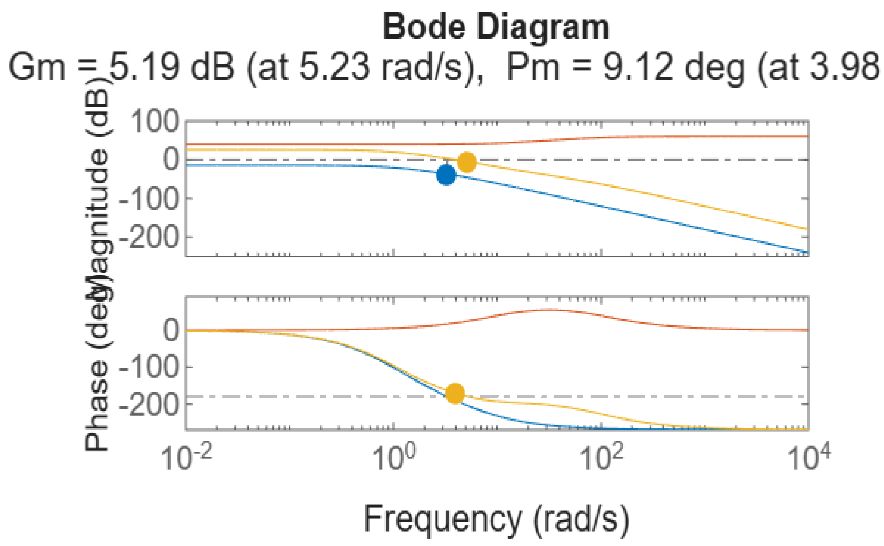


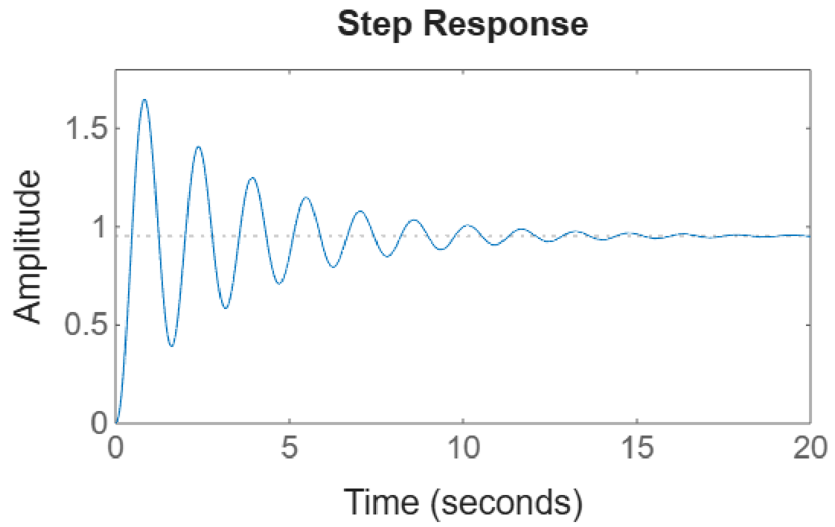

```
sz = -5; sp = -50;K_L = K_P*sp*(s-sz)/(s-sp)/sz; margin(K_L*H); step(feedback(K_L*H,1));
```





```
sz = -10; sp = -100;K_L = K_P*sp*(s-sz)/(s-sp)/sz; margin(K_L*H); step(feedback(K_L*H,1));
```






```
sz = -20; sp = -200;K_L = K_P*sp*(s-sz)/(s-sp)/sz; margin(K_L*H); step(feedback(K_L*H,1));
```