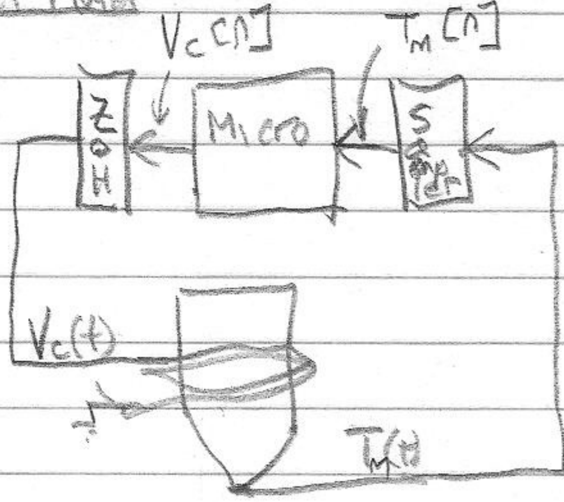
2/10/25 Blackboard Notes (4 Pages) Followed by Formatted Slides

2/10/25 6.310 ①

Today

- F.Q. Diff Eqns – Nat Freq, solns
- Application to prop. control
- stability & steady state

Thermal Model



$V_c = f(u)$

Time!

$$T_m[n] = T_M[n-1] + \Delta T \, \gamma_{th} \, u[n-1]$$

$$u[n] = K_p \left(T_d[n] - T_M[n] \right)$$

or

$$u[n] = K_{ff} T_d[n] + K_p \left(T_d[n] - T_M[n] \right)$$

(K_{ff}: feed forward; K_p: proportional feedback)

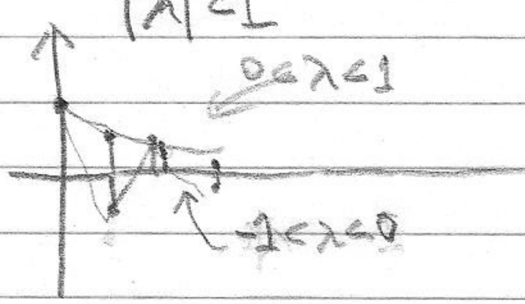
D. E. Solns

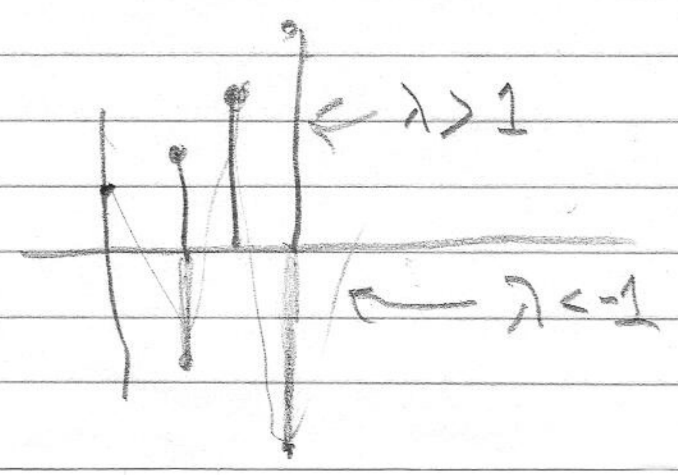
$$y[n] = \lambda y[n-1] + \gamma u[n-1]$$

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{n-1-m} u[m]$$

Spose $u[n] = 0$ $\quad y[0] = 1$

$|\lambda| < 1$





Steady State $u[n] = u_0 \quad \forall\ n$

$$y[n] = \lambda^n y[0] + \gamma\left(\sum_{m=0}^{n-1} \lambda^{n-1-m}\right) u_0$$

if $|\lambda| < 1$

$$y[\infty] = 0 + \frac{\gamma}{1-\lambda} u_0$$

or

$$y[\infty] = \lambda y[\infty] + \gamma u_0$$

$$y[\infty] = \frac{\gamma}{1-\lambda} u_0$$

Heater $K_{ff} = 0 \quad K_p \neq 0$

$$T_m[n] = \left(1 - \Delta T \gamma_{th} K_p\right) T_m[n-1] + \Delta T \gamma_{th} K_p T_d[n-1]$$

if $|1 - \Delta T \gamma_{th} K_p| < 1$

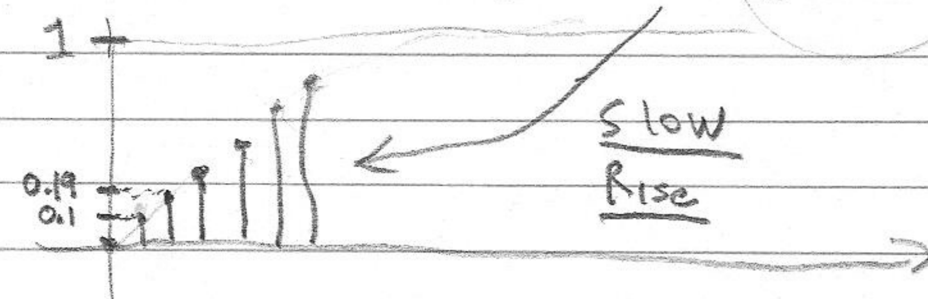
in S.S.
$$T[\infty] = \frac{\Delta T \gamma_{th} K_p}{1 - (1 - \Delta T \gamma_{th} K_p)} T_d[\infty]$$

$$= T_d[\infty]$$

Small K_p Example: $\Delta T \gamma_{th} K_p = 0.1 \quad T_d[n] = 1 \ \forall n$

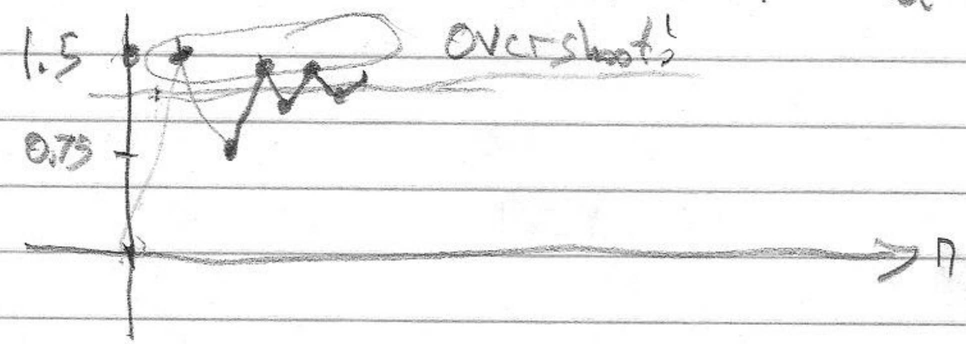
$$T_m[n] = 0.9\, T_m[n-1] + 0.1\, T_d[n-1]$$

$T_d[n-1] = 1$



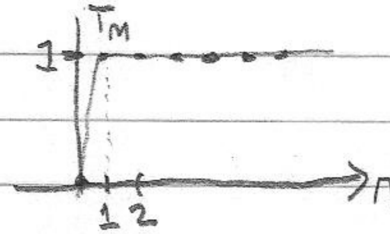
Make K_p Bigger $\Delta T \gamma_{th} K_p = 1.5$

$$T_m[n] = -0.5\, T_m[n-1] + 1.5\, T_d[n-1]$$



Make K_p Perfect $\Delta T \gamma_{th} K_p = 1$

$$T_m[n] = 1 \cdot T_d[n-1] \Rightarrow$$



Include Heat loss

$$T_m[n] = T_m[n-1] + \Delta T \left[\gamma_{th} K_p \left(T_d[n-1] - T_m[n-1] \right) - \beta\, T_m[n-1] \right]$$

$$T_m[n] = \left[1 - \Delta T \left(\gamma_{th} K_p + \beta \right) \right] T_m[n-1] + \Delta T\, \gamma_{th} K_p\, T_d[n-1]$$

S.S. $T_m[\infty] = \dfrac{\Delta T \gamma_{th} K_p}{1 - \left(1 - \Delta T(\gamma_{th} K_p + \beta)\right)} T_d[\infty]$

if $|(1 - \Delta T(\gamma_{th} K_p - \beta))| < 1$!

$$= \frac{\gamma_{th} K_p}{\gamma_{th} K_p + \beta} T_d[\infty]$$

Bigger K_p
Better Accuracy $= \left(1 - \dfrac{\beta}{\gamma_{th} K_p + \beta} \right) T_d[\infty]$

Summary

1) Bound on K_p for Stability

$$\lambda = 1 - \Delta T \gamma_{th} K_p \quad \begin{cases} \lambda = 1 & K_p = 0 \\ \lambda = 0 & K_p = \frac{1}{\Delta T \gamma_{th}} \\ \lambda = -1 & K_p = \frac{2}{\Delta T \gamma_{th}} \end{cases}$$

Solutions don't oscillate!

2) Bigger K_p, Better Disturbance rejection

$$T_m[\infty] = 1 - \frac{\beta}{\gamma_{th} K_p + \beta} T_d[\infty]$$

3) What about disturbing inputs Like in Lab?

Recap: General Form of First Order System

The general form of a first order DT system:

$$y[n] = \lambda y[n-1] + bx[n-1] \qquad (\#1)$$

Notes on the general form:

- Our goal is to solve for $y[n]$
- $x[n]$ is the input or driving function we set
- λ is the natural frequency
- b is a multiplicative constant

Recap: ZSR of First-Order DT System: Finding $y[n]$

We studied the case when $x[n] = 1$ for all $n \geq 0$ and $y[0] = 0$.

- This is known as the Zero State Response (ZSR)

We solved for $y[n]$ to obtain:

$$y[n] = \frac{b}{1-\lambda}(1 - \lambda^n).$$

In particular, we found that $y[n]$ converges to a finite value as $n \to \infty$ when $-1 < \lambda < 1$.

Generalizing to Arbitrary Inputs Signals

Our first-order difference equations have two convenient properties: linearity and time-invariance.

Linearity:

- If $x_a[n] \rightarrow y_a[n]$ and $x_b[n] \rightarrow y_b[n]$, then $Ax_a[n] + Bx_b[n] \rightarrow Ay_a[n] + By_b[n]$.

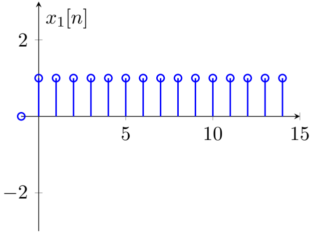
Time Invariance:

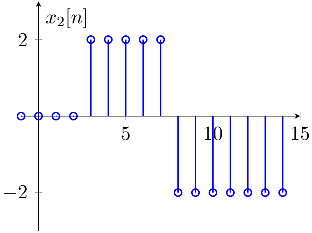
- If $x[n] \rightarrow y[n]$, then $x[n - n_0] \rightarrow y[n - n_0]$.

Here, A, B are constants, "$\rightarrow$" means "leads to," and n_0 is an integer-valued length of time.

Check Yourself: Defining a Complex Driving Function

Consider input signal $x_1[n]$ on the left and a more complex input $x_2[n]$ on the right:



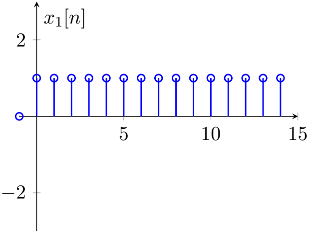


Define $x_2[n]$ in terms of rescaled and time-shifted $x_1[n]$ signals.

$$x_2[n] = ?$$

Check Yourself: Defining a Complex Driving Function

Consider input signal $x_1[n]$ on the left and a more complex input $x_2[n]$ on the right:

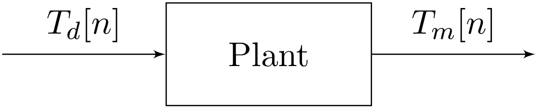


Define $x_2[n]$ in terms of rescaled and time-shifted $x_1[n]$ signals.

$$x_2[n] = 2x_1[n-3] - 4x_1[n-8]$$

Recall: Feedforward Control

Let's return to the idea of feedforward control:



We can analyze the feedforward controller:

$$\text{FF controller:} \quad u[n] = K_{ff} T_d[n],$$

$$\text{Plant:} \quad \frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma u[n-1].$$

Feedforward First Order DT System

For our feedforward system, we arrive at the following equation:

$$\frac{T_m[n] - T_m[n-1]}{\Delta T} = \gamma K_{ff} T_d[n-1].$$

Rearranging, we have:

$$T_m[n] = T_m[n-1] + \Delta T \gamma K_{ff} T_d[n-1].$$

What is our system's natural frequency? What will be its steady-state behavior?

Feedforward System's Steady-state Behavior

Comparing the general first order DT system with our result,

$$T_m[n] = T_m[n-1] + \Delta T \gamma K_{ff} T_d[n-1],$$

we can see that the natural frequency is $\lambda = 1$.

Without any feedback control, this system is unstable and likely will not perform very well.

- However, this is not the end of the story for feedforward control!

Recall: Choosing K_p for Stability

At the end of last lecture, we analyzed our first order DT system for our system with feedback:

$$T_m[n] = (1 - \gamma \Delta T K_p) T_m[n-1] + \gamma \Delta T K_p T_d[n-1].$$

Comparing this result with the general first order DT system, we found that we need,

$$\begin{gathered} -1 < \lambda < 1, \\ -1 < 1 - \gamma \Delta T K_p < 1, \\ 0 < K_p < \frac{2}{\gamma \Delta T}, \end{gathered}$$

to guarantee a stable system.

Towards a "Realistic" Controller

Our old plant equation is given by:

$$T_m[n] = T_m[n-1] + \Delta T \gamma u[n-1].$$

Realistically, there are other environmental factors that effect our plant. We can add another term in the equation:

$$T_m[n] = T_m[n-1] + \Delta T \gamma u[n-1] - \Delta T \beta T_m[n-1].$$

Here, $\beta \geq 0$ is a constant relating heat loss to the instantaneous temperature $T_m[n]$.

Proportional Controller for Plant with Loss

With this system, we can implement the same proportional feedback controller:

$$u[n] = K_p(T_d[n] - T_m[n]).$$

The system equation becomes:

$$T_m[n] = (1 - \gamma \Delta T K_p - \Delta T \beta) T_m[n-1] + \gamma \Delta T K_p T_d[n-1].$$

Note that we have a new term $-\Delta T \beta T_m[n-1]$, which changes our selection of K_p.

Stability of System with Loss

Our system with loss is still a first-order DT system and we can analyze the stability in the same way:

$$
\begin{aligned}
-1 < \lambda < 1, \\
-1 < 1 - \gamma \Delta T K_p - \Delta T \beta < 1, \\
\frac{-\beta}{\gamma} < K_p < \frac{2 - \beta \Delta T}{\gamma \Delta T}.
\end{aligned}
$$

Choosing a value of K_p within this range guarantees stability.

Convergence of System with Loss

Suppose we want our system to converge to a steady state value as quickly as possible. As before, we can set the natural frequency $\lambda = 0$:

$$\lambda = (1 - \gamma K_p \Delta T - \Delta\beta) = 0.$$

Solving for K_p, we obtain:

$$K_p = \frac{1 - \Delta T \beta}{\gamma \Delta T}.$$

This analysis yields a K_p that is optimal with respect to convergence speed. However, there are other factors to consider...

Steady-State Error with Loss

Let's calculate the steady-state error. We'll define the error term as:

$$e[n] = T_d[n] - T_m[n].$$

Our goal is to find $e[\infty] = \lim_{n\to\infty} e[n]$.

We can rearrange the system equation as:

$$\begin{aligned} T_m[n] &= (1 - \gamma K_p \Delta T - \Delta T \beta) T_m[n-1] + \gamma \Delta T K_p T_d[n-1] \\ e[n] &= \underbrace{(1 - \gamma K_p \Delta T - \Delta T \beta)}_{\lambda} e[n-1] + \Delta T \beta T_d[n-1]. \end{aligned}$$

Thus, as n approaches infinity, we obtain;

$$e[\infty] = \lambda e[\infty] + \Delta T \beta T_d[\infty] \Rightarrow e[\infty] = \frac{\Delta T \beta T_d[\infty]}{1 - \lambda}.$$

Nonzero Steady-State Error!

Our steady-state error is $e[\infty] = \frac{\Delta T \beta T_d[\infty]}{1-\lambda}$.

- In particular, as long as $\beta \neq 0$, our control system will have a steady-state error!
- In many realistic situations, there is no solution that optimizes every aspect of the control system.
- Prioritizing faster convergence vs. small steady-state error is a design choice.

Can we design a new controller that removes the steady-state error?

Combination Feedforward-and-Proportional Controller

Let's define a new controller as:

$$u[n] = \underbrace{K_{ff}T_d[n]}_{\text{feedforward}} + \underbrace{K_p(T_d[n] - T_m[n])}_{\text{feedback}}.$$

Now, we have 2 different gains to choose: K_{ff} and K_p. Our system equation becomes:

$$T_m[n] = (1 - \gamma K_p \Delta T - \Delta T \beta)T_m[n-1] + \gamma \Delta T(K_p + K_{ff})T_d[n-1].$$

What impact does picking K_p, K_{ff} have on the steady-state error of this system?

Computing Steady-State Error

Recall that we can define an error signal $e[n] = T_d[n] - T_m[n]$. We can rewrite our system equation as:

$$\begin{aligned} T_m[n] &= (1 - \gamma K_p \Delta T - \Delta T \beta) T_m[n-1] + \gamma \Delta T (K_p + K_{ff}) T_d[n-1], \\ e[n] &= \underbrace{(1 - \gamma K_p \Delta T - \Delta T \beta)}_{\lambda} e[n-1] + (-\gamma K_{ff} + \beta) \Delta T \ T_d[n-1], \end{aligned}$$

$$\Rightarrow e[n] = \lambda e[n-1] + (-\gamma K_{ff} + \beta) \Delta T \ T_d[n-1].$$

Computing Steady-State Error

Now, the steady-state error becomes:

$$e[\infty] = \lambda e[\infty] + (-\gamma K_{ff} + \beta)\Delta T T_d[\infty],$$
$$\Rightarrow e[\infty] = \frac{(-\gamma K_{ff} + \beta)\Delta T T_d[\infty]}{1-\lambda}.$$

Can we make the steady-state error $e[\infty] = 0$? Yes!

We can set $K_{ff} = \frac{\beta}{\gamma}$. In the second part of Lab 1, we'll see how to compute β, γ analytically.