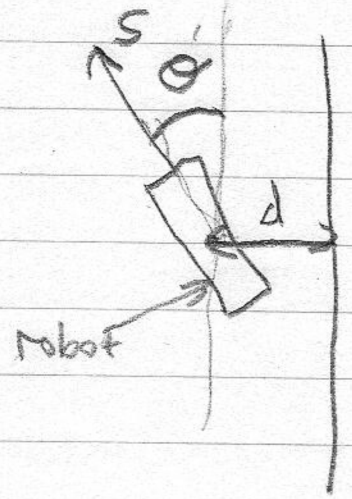
2/19/25 6.3100/2 (1)

Reminder - Path Following Robot



d = dist. from line
S = Forward Speed
θ = robot angle
ΔT = time btwn samples
ω = angular velocity (controllable)

Eqns
$$d[n] = d[n-1] + S\Delta T \sin\theta \quad (\approx \theta \text{ In radians!})$$
$$\theta[n] = \theta[n-1] + \Delta T\, \omega[n-1]$$

Proportional Control
$$\omega[n] = K_p (d_d[n] - d[n])$$
desired distance to line ($d_d[n]$), measured distance ($d[n]$)

If $d_d[n] = 0$ (follow the line)

$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & S\Delta T \\ -K_p\Delta T & 1 \end{bmatrix}}_{A} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}$$

$$\text{evals}(A) = \lambda_1, \lambda_2 = 1 \pm j \Delta T \sqrt{KS}$$

$$d[n] = c_1 \lambda_1^n + c_2 \lambda_2^n$$

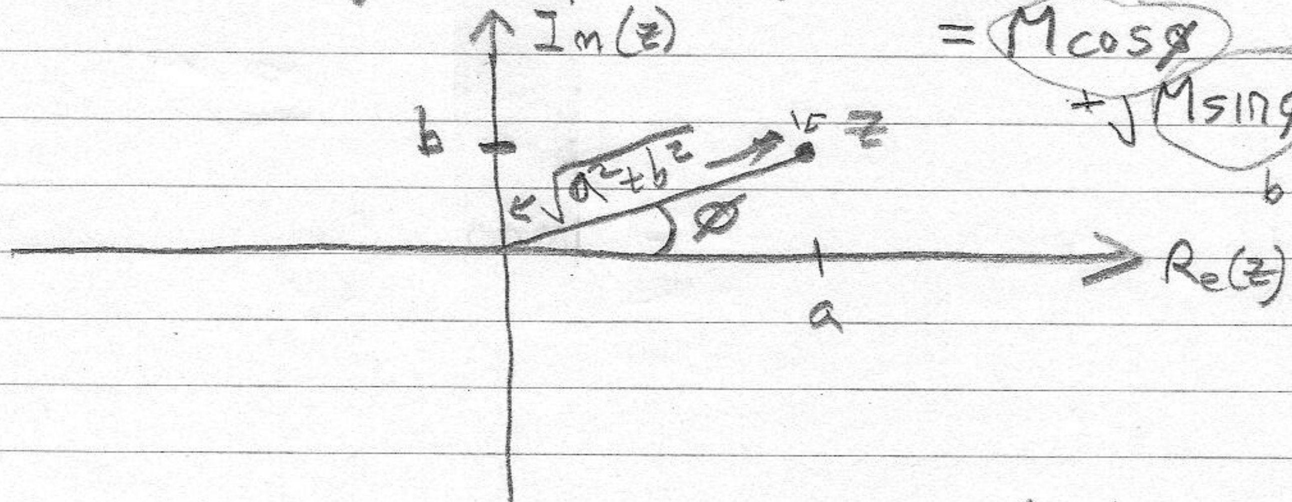
If $d[0] = 1$, $\theta[0] = 0 \Rightarrow d[1] = 1$ $\left(A \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$\Rightarrow c_1\lambda_1^0 + c_2\lambda_2^0 = 1 = c_1 + c_2$ $\quad c_1\lambda_1^1 + c_2\lambda_2^1 = 1$

Polar Form

$$z = a + bj \text{ (complex \#)} = M e^{j\phi} = M\cos\phi + j M\sin\phi$$

($M = \sqrt{a^2+b^2}$; $M\cos\phi$ → a; $M\sin\phi$ → b)



$$a = M\cos\phi \Rightarrow \phi = \arccos\left(\frac{a}{M}\right)$$
$$b = M\sin\phi \Rightarrow \phi = \arcsin\left(\frac{b}{M}\right)$$

$$z^n = \left(M e^{j\phi}\right)^n = M^n e^{j\phi n} = M^n\cos n\phi + jM^n \sin n\phi$$

Back to $d[n] = C_1\lambda_1 + C_2\lambda_2$

($d[n]$ ↑ real; λ_1, λ_2: complex but complex conjs (roots of real poly); C_1, C_2: complex conj)

Therefore $\text{Im}(C_1\lambda_1) = -\text{Im}(C_2\lambda_2)$

$$C_1 = M_c e^{j\phi_c} \qquad C_2 = M_c e^{-j\phi_c}$$

$$C_1\lambda_1^n = M_{c_1} M_\lambda^n e^{j(\phi_{c_1} + n\phi_{\lambda_1})} \qquad C_2\lambda_2^n = M_{c_2} M_{\lambda_2}^n e^{j(\phi_{c_2} - n\phi_{\lambda_2})}$$

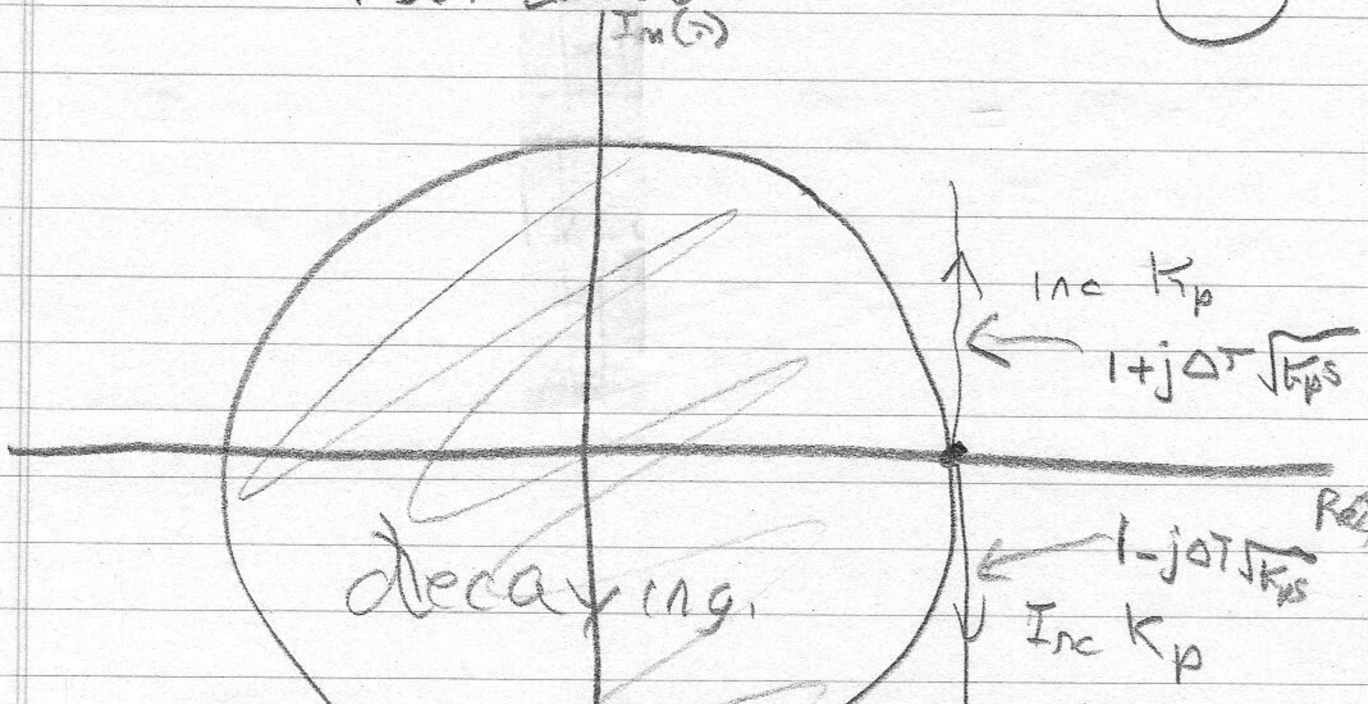
For imaginary parts to cancel

$$\phi_{c_1} = -\phi_{c_2} \qquad M_{c_1} = M_{c_2}$$

$$d[n] = M_c M_\lambda^n \cos(\phi_c + n\phi_\lambda)$$

(shift → ϕ_c; period → $n\phi_\lambda$)

Root Locus



intuition ⇒

$d[n]=0$ but $\theta[n]\neq 0$

so ↓

Drive angle to zero

Suppose

$$w[n] = K_p(\underbrace{d_d[n]}_{=0\ \forall n} - d[n]) + K_\theta(-\theta[n])$$

$$\begin{pmatrix} d[n] \\ \theta[n] \end{pmatrix} = \begin{bmatrix} 1 & \Delta T S \\ -\Delta T K_p & 1-\Delta T K_\theta \end{bmatrix} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}$$

evals $\quad (\lambda-1)(\lambda-(1-\Delta T K_\theta)) + \Delta T^2 K_p S = 0$

$$\lambda_1, \lambda_2 = 1 - \frac{\Delta T K_\theta}{2} \pm \Delta T\sqrt{\frac{K_\theta^2}{4} - K_p S}$$

Hmm

Suppose we can't measure the angle

$$\theta[n] \cong \frac{d[n] - d[n-1]}{\Delta T} \approx \text{derivative}$$

So

$$\omega[n] = K_p\left(-d[n]\right) + K_d\left(-\left(\frac{d[n] - d[n-1]}{\Delta T}\right)\right)$$

(K_d: derivative gain)

$$\begin{bmatrix} d[n] \\ d[n-1] \\ \theta[n] \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta T S \\ 1 & 0 & 0 \\ -(K_p \Delta T + K_d) & K_d & 1 \end{bmatrix} \begin{bmatrix} d[n-1] \\ d[n-2] \\ \theta[n] \end{bmatrix}$$

$$\begin{aligned} \theta[n] &= \theta[n-1] + \Delta T \omega[n-1] \\ &= \theta[n-1] + \Delta T K_p(-d[n-1]) \\ &\quad + \Delta T K_d\left(-\frac{d[n-1] - d[n-2]}{\Delta T}\right) \end{aligned}$$

matlab script link:

https://introcontrol.mit.edu/_static/spring25/lectures/rootPlotLec2SS.m.zip

```matlab
function rootPlotLec2SS(Kp0, Kp1, deltaD, Kd0, Kd1, Ki0, Ki1)
% Plot Step responses and natural frequencies for line following Robot
% as Kp varies from Kp0->Kp1, (Kp0 > 0, Kp1 >= Kp0)
% deltaD (true,false) False: Use angle, True = use distance differences.
% optional Kd0,Kd1: Kd varies from Kd0->Kd1
% optional Ki0,Ki1: Ki varies from Ki0->Ki1
% NOTE!! Nat freqs corresponding maximum K's marked with BLACK star

% Zoom into neighborhood of 1 on nat Freqs plot
zoomThresh = 0; % 0.99; %0.99; %0.95;   % Set to zero for no zoom
S = 5;  % Robot Speed

% Zero out unset inputs
if nargin < 5; Kd0 = 0.0; Kd1 = 0.0; end
if nargin < 7; Ki0 = 0.0; Ki1 = 0.0; end

% Error check input
assert(Kp1 >= Kp0);
assert(Kd1 >= Kd0);
assert(Ki1 >= Ki0);

% Determine DT system order (size of matrix A)
order = 2;
if(Kd1 > 0);  order = order + deltaD; end
if(Ki1 > 0);  order = order + 1; end

% Should not need to change these
dT = 1.0e-1;    % Time between samples
seconds = 5;    % Response plot time interval
nParamVals = 1000;  % Number of param value points

% Close existing plots
close all;

% Function that computes system matrix A and eig(A)=natfreqs
    function [natFreqs,A] = getNatFreqs(Kp,Kd,Ki)
        % Gets A matrix and nat freqs, assumes first state is theta
        if order <= 2
            A(1,:) =  [1        S*dT];               % row one
            A(2,:)  = [-dT*Kp 1-dT*Kd];                % row two
        elseif order == 3
            if Ki == 0  % using Kp*d + Kd*deltad
                A(1,:) = [1                 0    S*dT];  % row one
                A(2,:) = [1                 0    0];    % row two
                A(3,:) = [-(dT*Kp+Kd)      Kd   1];    % row three
            else  % using Kp*d+Kd*theta+Ki*sumd
                A(1,:) = [1   0   S*dT];        % row one
                A(2,:) = [0   0      0];        % row two
                A(3,:) = [0   0      0];        % row three
            end
        else % Use Kp*d + Kd*deltad + Ki*sumd
            % Fix the A matrix!
            A(1,:) = [1   0   S*dT  0];        % row one
            A(2,:) = [1   0   0     0];        % row two
            A(3,:) = [0   0   0     0];        % row three
            A(4,:) = [0   0   0     0];        % row four
        end
        natFreqs = eig(A);
    end
```

```
[natFreqs,A0] = getNatFreqs(Kp0,Kd0,Ki0);
[~,A1] = getNatFreqs(Kp1,Kd1,Ki1);

% Preallocate vectors for time and state
T = 0:ceil(seconds/dT);
X = zeros(size(A0,1),length(T));
X(1,1) = 1;  % Set initial distance to 1, other initial states to zero.

p1 = figure(1);
p1.Position(1) = 0.5*p1.Position(1);
% Plot ZIR's
for plt = 1:2
    if plt == 1
        A = A0; Kp = Kp0; Kd = Kd0; Ki = Ki0;
    else
        A = A1; Kp = Kp1; Kd = Kd1; Ki = Ki1;
    end
    for i = 2:length(T)
        X(:,i) = A*X(:,i-1);
    end
    subplot(2,1,plt);
    plot(T,X(1,:));
    xlabel('Time in Samples');
    title(['ZIR',',Kp=',num2str(Kp),',Kd=',num2str(Kd),',Ki=',num2str(Ki)]);
end


% Get an array # natural frequency rows, # param values column
natMat = zeros(size(natFreqs,1),nParamVals);
for i = 1:nParamVals
    Kdl = Kd0 + (Kd1-Kd0)*i/nParamVals;
    Kpl = Kp0 + (Kp1-Kp0)*i/nParamVals;
    Kil = Ki0 + (Ki1-Ki0)*i/nParamVals;
    natMat(:,i) = getNatFreqs(Kpl,Kdl,Kil);
end

%Screen out small magnitude nat freqs.
rootsLarge = natMat(abs(natMat) >= zoomThresh);

p2 = figure(2);
p2.Position(1) = p2.Position(1)+0.7*p1.Position(3);

hold on;
```

```
% Plot trajectories of slower natural frequencies
for rootType = 1:3
    if(rootType == 1)
        roots = rootsLarge(imag(rootsLarge) == 0); pStr = 'mo';
    elseif(rootType == 2)
        roots = rootsLarge(imag(rootsLarge) > 0); pStr = 'ro';
    else
        roots = rootsLarge(imag(rootsLarge) < 0); pStr = 'go';
    end
    plot(real(roots), imag(roots),pStr);
end
if length(natMat) > 4
    last4 = natMat(:,end-4:end);
    last4 = last4(abs(last4) >= zoomThresh);
    plot(real(last4), imag(last4),'k*');
end

% Plot entire (or fraction of if zoom true) of the unit circle
if zoomThresh > 0
    maxAngle = max(abs(angle(rootsLarge(:))));
else
    maxAngle = pi;
end
ang = linspace(-1.5*maxAngle, 1.5*maxAngle, 30000);
circlepts = cos(ang)+ 1j*sin(ang);
plot(real(circlepts), imag(circlepts), 'b.');
hold off;

% Give the set of plots a title
title(['unitCirc(Blue)',...
    ', Kp:',num2str(Kp0),'->',num2str(Kp1),...
    ', Kd:',num2str(Kd0),'->',num2str(Kd1),...
    ', Ki:',num2str(Ki0),'->',num2str(Ki1)]);
end
```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Time in Samples
ZIR,Kp=5,Kd=0,Ki=0
Time in Samples
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unitCirc(Blue), Kp:0->5, Kd:0->0, Ki:0->0
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Command Window
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Unrecognized function or variable 'l10'.
Did you mean:
>> rootPlotLec2SS(10, 10, true, 0, 1)
>> rootPlotLec2SS(10, 10, true, 0, 5)
>> rootPlotLec2SS(0, 10, true, 2, 2)
>> rootPlotLec2SS(0, 10, true, 4, 4)
>> rootPlotLec2SS(0, 5)
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>> rootPlotLec2SS(0, 5)
>> rootPlotLec2SS(0, 10, true, 4, 4)
>> rootPlotLec2SS(5, 5, false, 0, 4)
>> rootPlotLec2SS(5, 5, false, 0, 6)
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ZIR,Kp=5,Kd=10,Ki=0
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>> rootPlotLec2SS(0, 5)
>> rootPlotLec2SS(0, 10, true, 4, 4)
>> rootPlotLec2SS(5, 5, false, 0, 4)
>> rootPlotLec2SS(5, 5, false, 0, 6)
>> rootPlotLec2SS(5, 5, false, 0, 8)
>> rootPlotLec2SS(5, 5, false, 0, 10)
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ZIR,Kp=5,Kd=12,Ki=0
Time in Samples
Figure 2
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unitCirc(Blue), Kp:5->5, Kd:0->12, Ki:0->0
Command Window
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>> rootPlotLec2SS(5, 5, false, 0, 4)
>> rootPlotLec2SS(5, 5, false, 0, 6)
>> rootPlotLec2SS(5, 5, false, 0, 8)
>> rootPlotLec2SS(5, 5, false, 0, 10)
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>> rootPlotLec2SS(5, 5, false, 0, 12)
>> shg
>> rootPlotLec2SS(5, 5, false, 0, 12)
>>
d[n−1],


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ZIR,Kp=2,Kd=6,Ki=0
Time in Samples
ZIR,Kp=20,Kd=6,Ki=0
Time in Samples
Figure 2
unitCirc(Blue), Kp:2->20, Kd:6->6, Ki:0->0
Command Window
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>> rootPlotLec2SS(5, 5, false, 0, 10)
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>> rootPlotLec2SS(5, 5, false, 0, 12)
>> shg
>> rootPlotLec2SS(5, 5, false, 0, 12)
>> rootPlotLec2SS(0, 20, false, 6, 6)
>> rootPlotLec2SS(1, 20, false, 6, 6)
>> rootPlotLec2SS(2, 20, false, 6, 6)
>>
d[n-1],


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ZIR,Kp=2,Kd=6,Ki=0
Time in Samples
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ZIR,Kp=2,Kd=2,Ki=0
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>> shg
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>> rootPlotLec2SS(2, 20, true, 6, 6)
>> rootPlotLec2SS(2, 2, true, 0, 6)
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>>
d[n-1],


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Figure 1
ZIR,Kp=2,Kd=0,Ki=0
Time in Samples
ZIR,Kp=2,Kd=1,Ki=0
Time in Samples
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unitCirc(Blue), Kp:2->2, Kd:0->1, Ki:0->0
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>> rootPlotLec2SS(5, 5, false, 0, 12)
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>> rootPlotLec2SS(2, 2, true, 0, 2)
>> rootPlotLec2SS(2, 2, true, 0, 1)
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d[n-1],