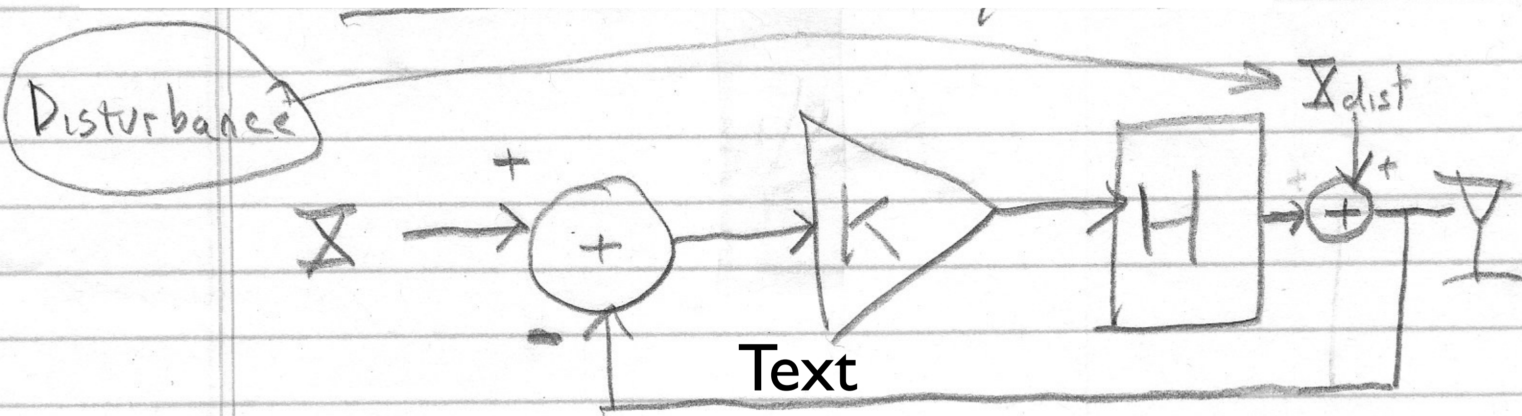
Design Examples 6.3100/2, Mon. 4/7/25



$\underline{X_{dist} = 0}$

$$Y = G X \qquad G(s) = \frac{K(s)H(s)}{1 + K(s)H(s)}$$

$\underline{X = 0}$

$$Y = KH(\underbrace{X}_{=0} - Y) + X_{dist}$$

$$Y = \underbrace{\frac{1}{1 + KH}}_{G_{dist}} X_{dist}$$

$\underline{\text{If}}$ $|K(j\omega)H(j\omega)| >> 1$

$G(j\omega) \simeq 1 \Leftarrow$ Good Tracking

$G_{dist}(j\omega) \simeq 0 \leftarrow$ Good Disturbance Rejection

If $K(j\omega)H(j\omega) \simeq -1$

$$G(j\omega) = \frac{\overset{KH}{-1}}{1+(-1)_{KH}} \Rightarrow \infty$$

$$G_{dist}(j\omega) = \frac{1}{1+(-1)} \Rightarrow \infty$$

So if $|K(j\omega)H(j\omega)| = 1$ and $\measuredangle K(j\omega)H(j\omega) = \pm 180°$

Tracking & Disturbance rejection are terrible!

And the closer $K(j\omega)H(j\omega)$ is to -1 the worse the tracking and disturbance rejection

But Just because $K(j\omega)H(j\omega)$ is always far from -1, no guarantee of stability for G

Sinusoidal Steady State exists only if system is stable

Why? Given $x(t) = e^{j\omega t}$

$y(t) = G(j\omega)e^{j\omega t}$ for large t only if G is stable

$G(j\omega)$ is physically meaningless if G is unstable

Example

Good phase margin necessary but not sufficient!

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$$H_{prop} = \frac{200}{(s+10)(s)(s)} \qquad K = 1$$

$$G = \frac{KH}{1+KH}$$

$$= \frac{200}{(s+10)(s)(s) + 200}$$

$$= \frac{200}{s^3 + 10s^2 + 200}$$

At $\omega = 4.3$ rad/sec

$$|KH_{prop}(j\omega)| = \left| \frac{200}{(j\omega+10)(j\omega)(j\omega)} \right|$$

$$= \left| \frac{200}{-j\omega^3 - 10\omega^2} \right| = \left| \frac{200}{-185-80j} \right|$$

$$\cong 1$$

④B

$$\angle KH(j\omega)\Big|_{\omega = 4.3\text{ rad/sec}} = \angle\frac{200}{(-185-80j)} \doteq -203^\circ \neq -180^\circ$$

$(\doteq \omega_{unity})$

at $\omega = 4.3$ $\quad \underbrace{|KH(j\omega)| \doteq 1 \quad \angle KH(j\omega) = -203^\circ}$

~~Satis~~ Satisfies phase margin conditions $\longrightarrow KH \neq -1$

<u>But</u> poles$(G(s))$ are unstable!

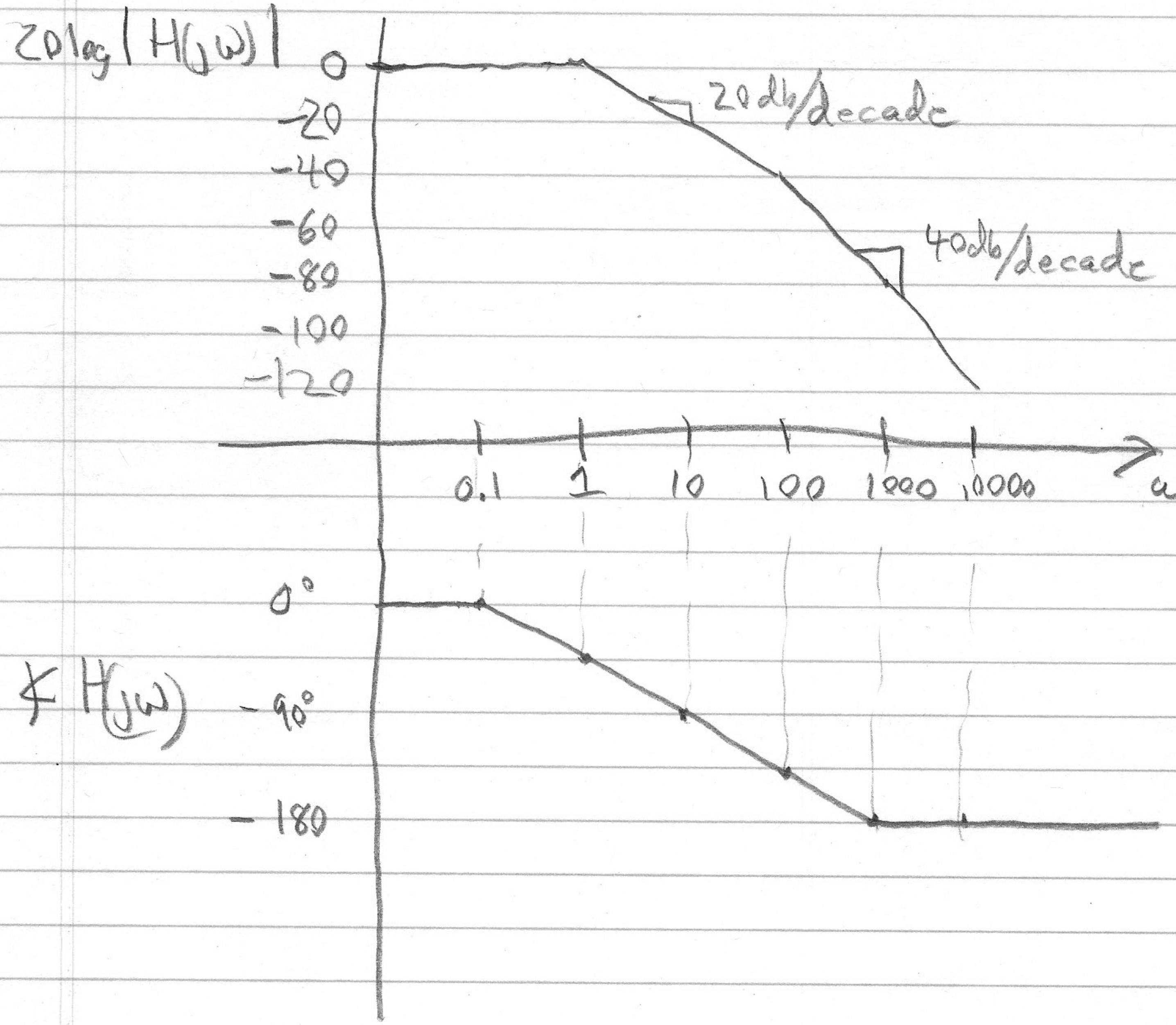
$$\text{poles}(G) = \underbrace{0.75}_{\substack{\text{real}\\\text{postive}}} \pm 4.1j, \ -11.5$$

$G(s)$ is unstable

Good phase margin is a <u>necessary</u> condition for good control <u>not a sufficient condition</u>!

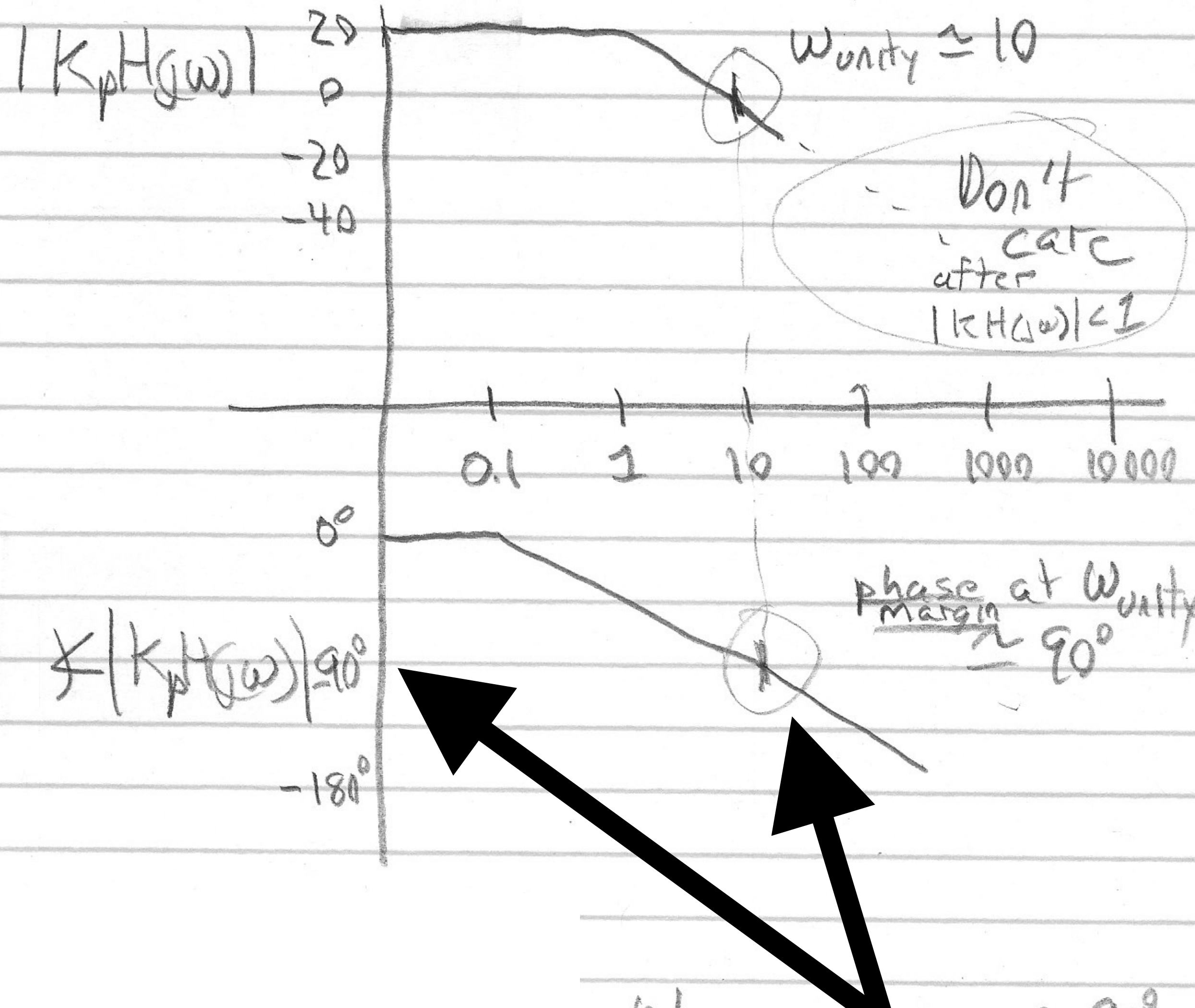
Example

$$H(s) = \frac{100}{(s+1)(s+100)}$$



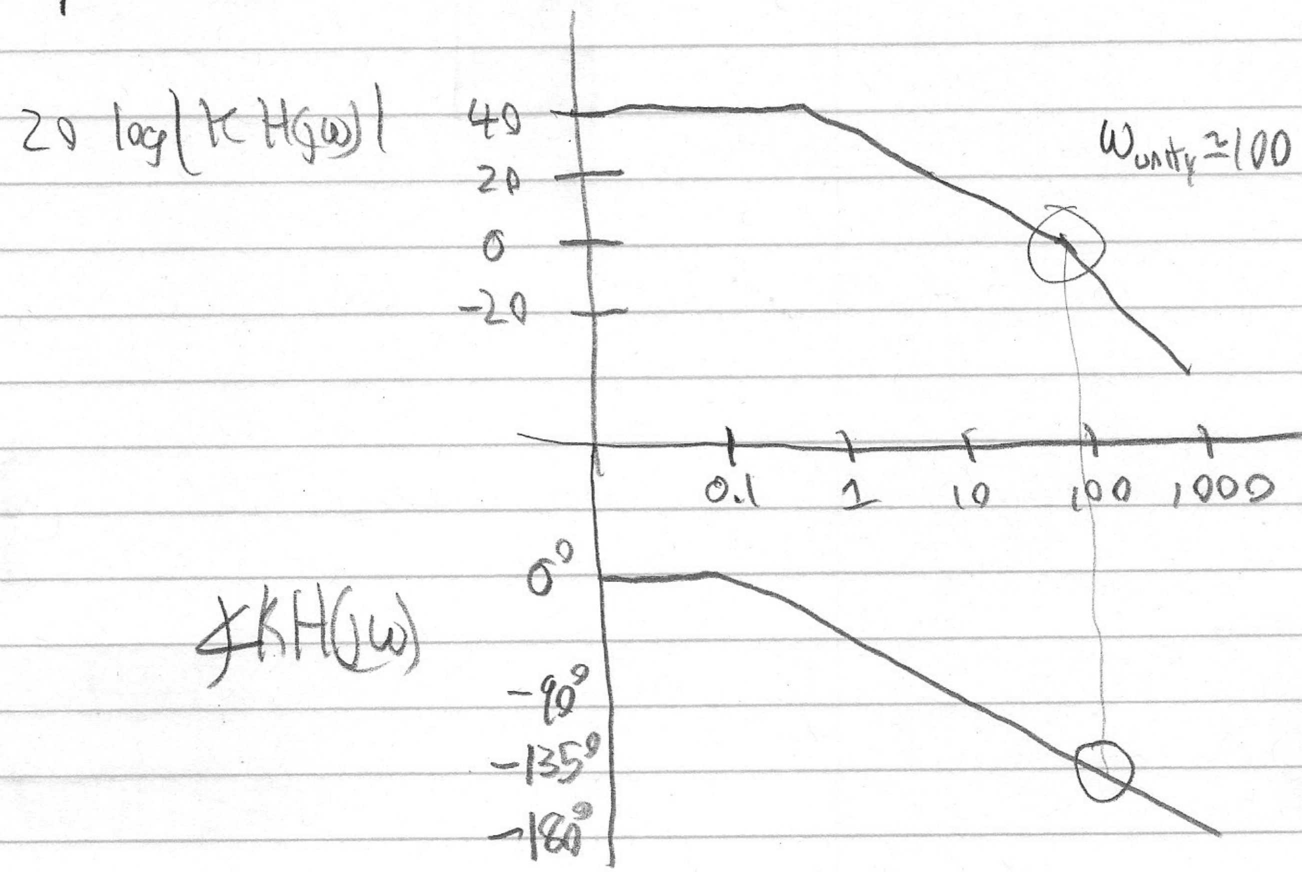
Design controller using $K_p = 10,\ 100\ \&\ 10{,}000$
What is the phase margin?

$K_p = 10$



phase margin ≃ 90°

$K_p = 100$



For $K_p = 100$ phase margin $\simeq 45^\circ$
okay but could be
better. Lead? Lag?

Idea 1 Place a lead compensator
with sz=q $= -100$, sp=p $= -1000$

Return to Later!

$$K(s) = K_p \frac{p}{q}\left(\frac{s-q}{s-p}\right) = K_p 10 \frac{s+100}{s+1000}$$

$$K(s)H(s) = 100 \cdot \frac{100}{(s+1)(s+100)} \cdot \frac{(s+100)}{s+1000} = \frac{10\,000}{(s+1)(s+1000)}$$

But Pole-Zero Cancellation Problematic!

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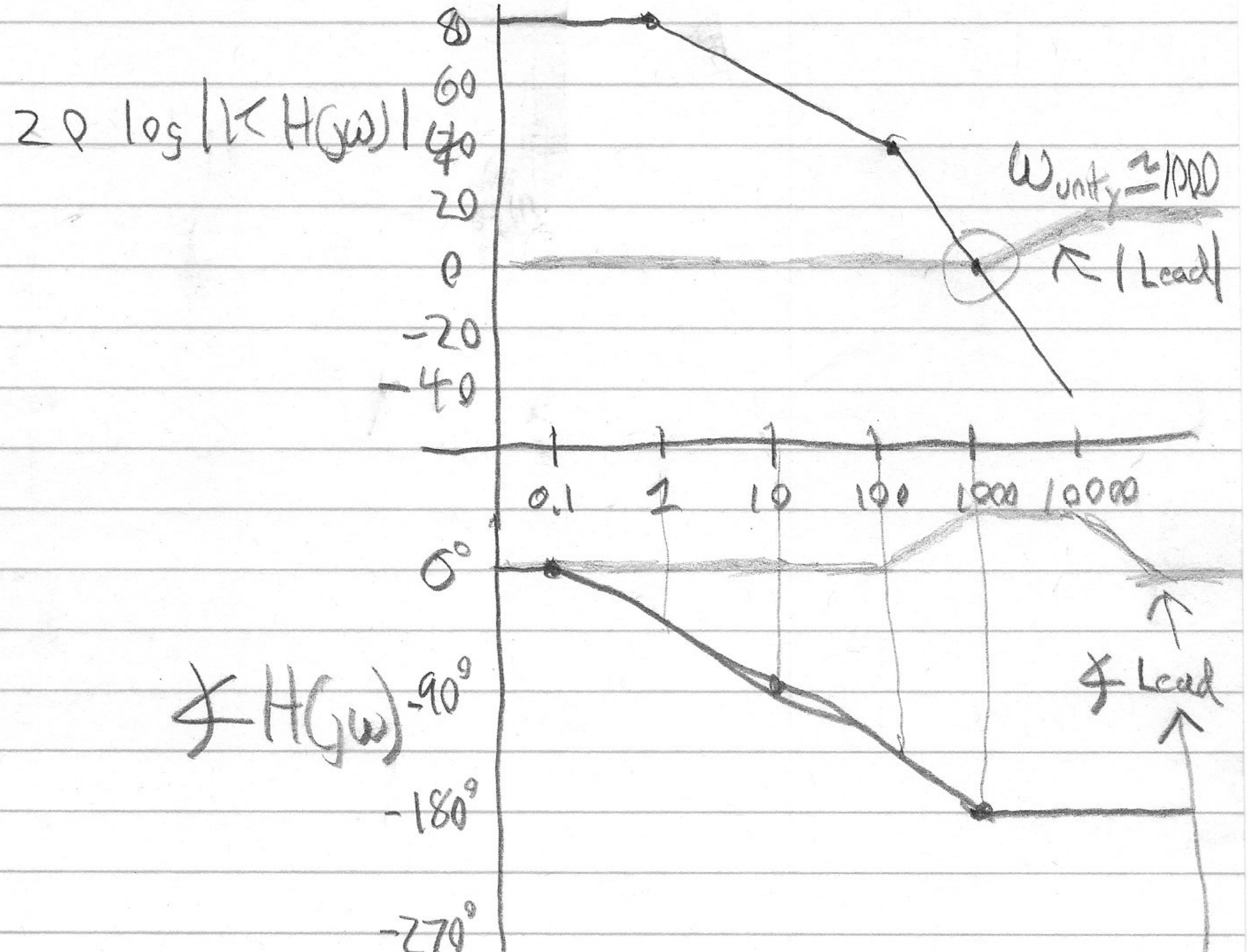
$\underline{KH >> 1}$!
Good Tracking !
Good dist Reject !

$K_p = 10000$

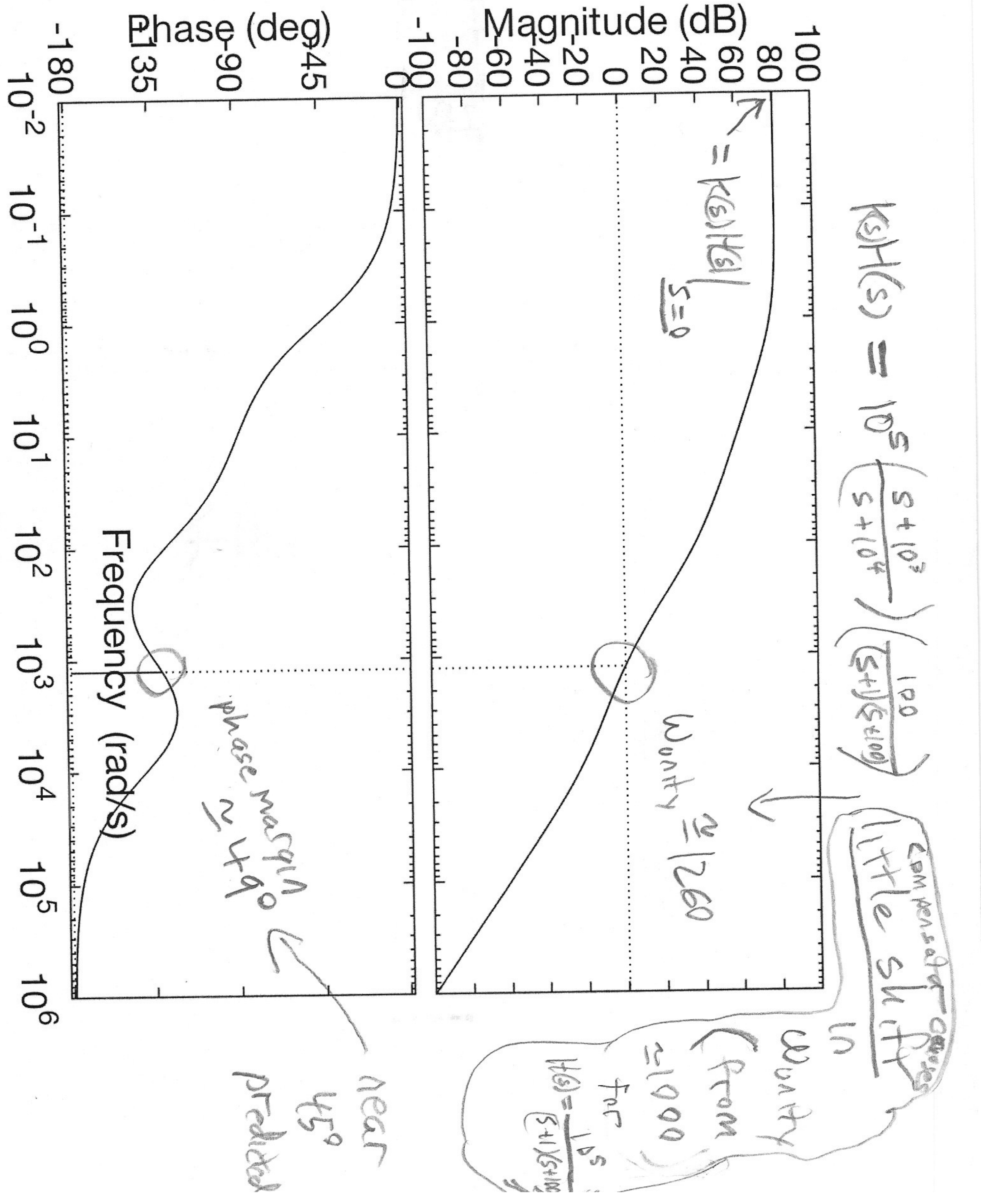


For K=10000 phase margin ≈ 0°
Bad, almost unstable

Idea 1 Place a lead with
$q = -1000 \quad p = -10000$

$$K(s)H(s) = 100{,}000\left(\frac{s+1000}{s+10000}\right)\left(\frac{100}{(s+1)(s+100)}\right)$$

Actual Margin

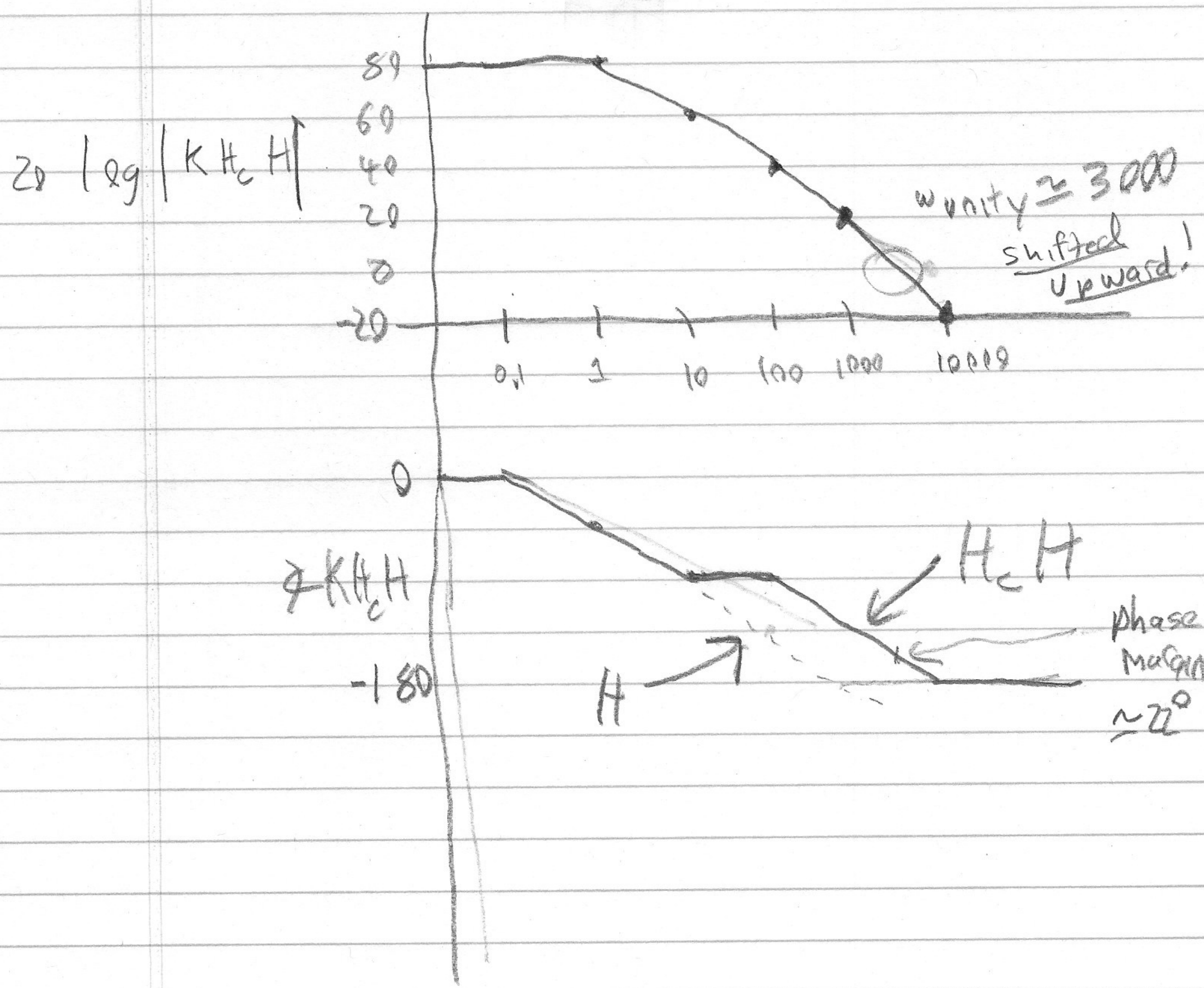


$\underline{K = 10,000}$

Too Soon

$$H_c(s) = 10\left(\frac{s+100}{s+1000}\right)$$

Zero at too low a frequency



Actual Too Soon

$H_c(s) = 10\,\frac{s+100}{s+1000}$

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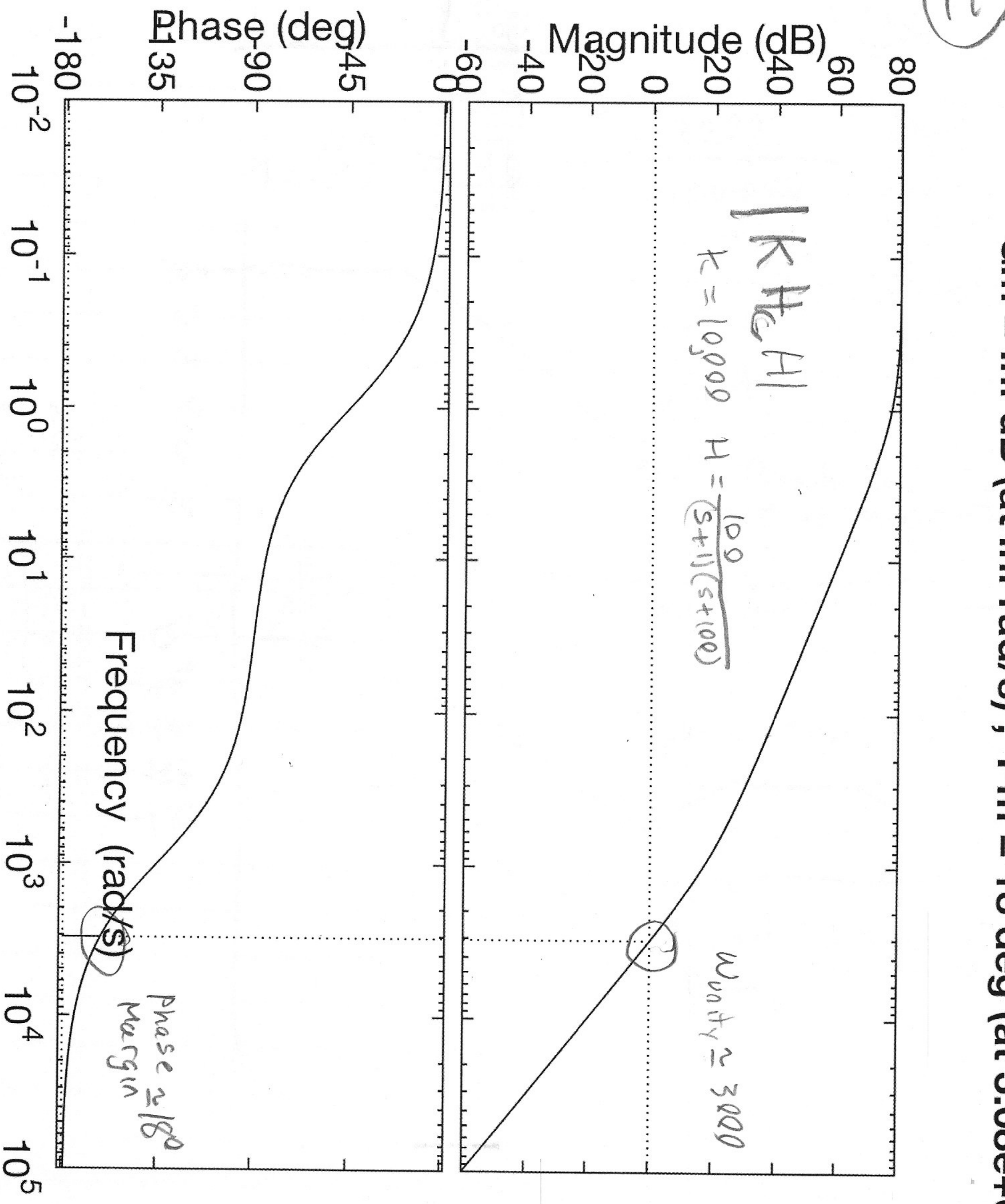
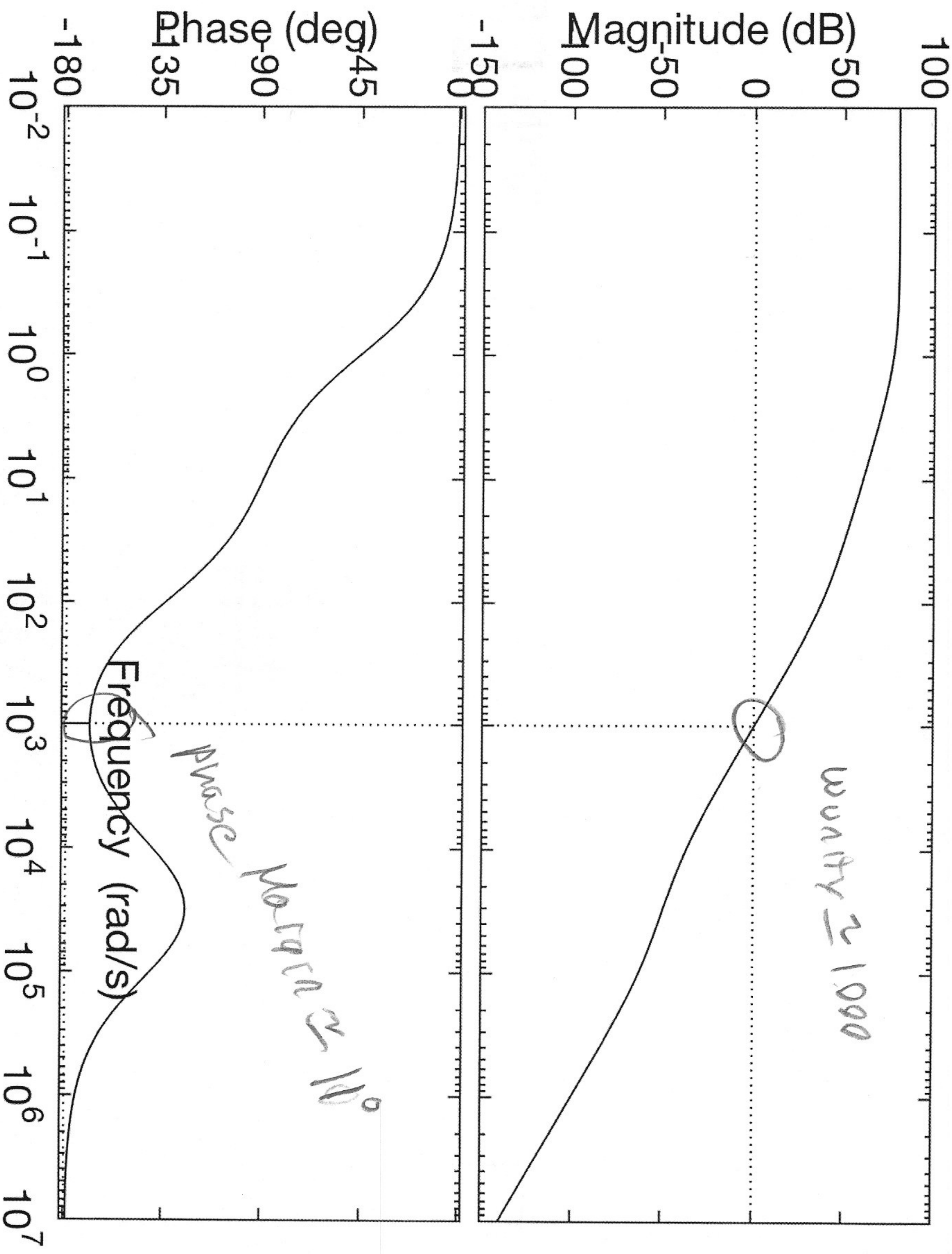
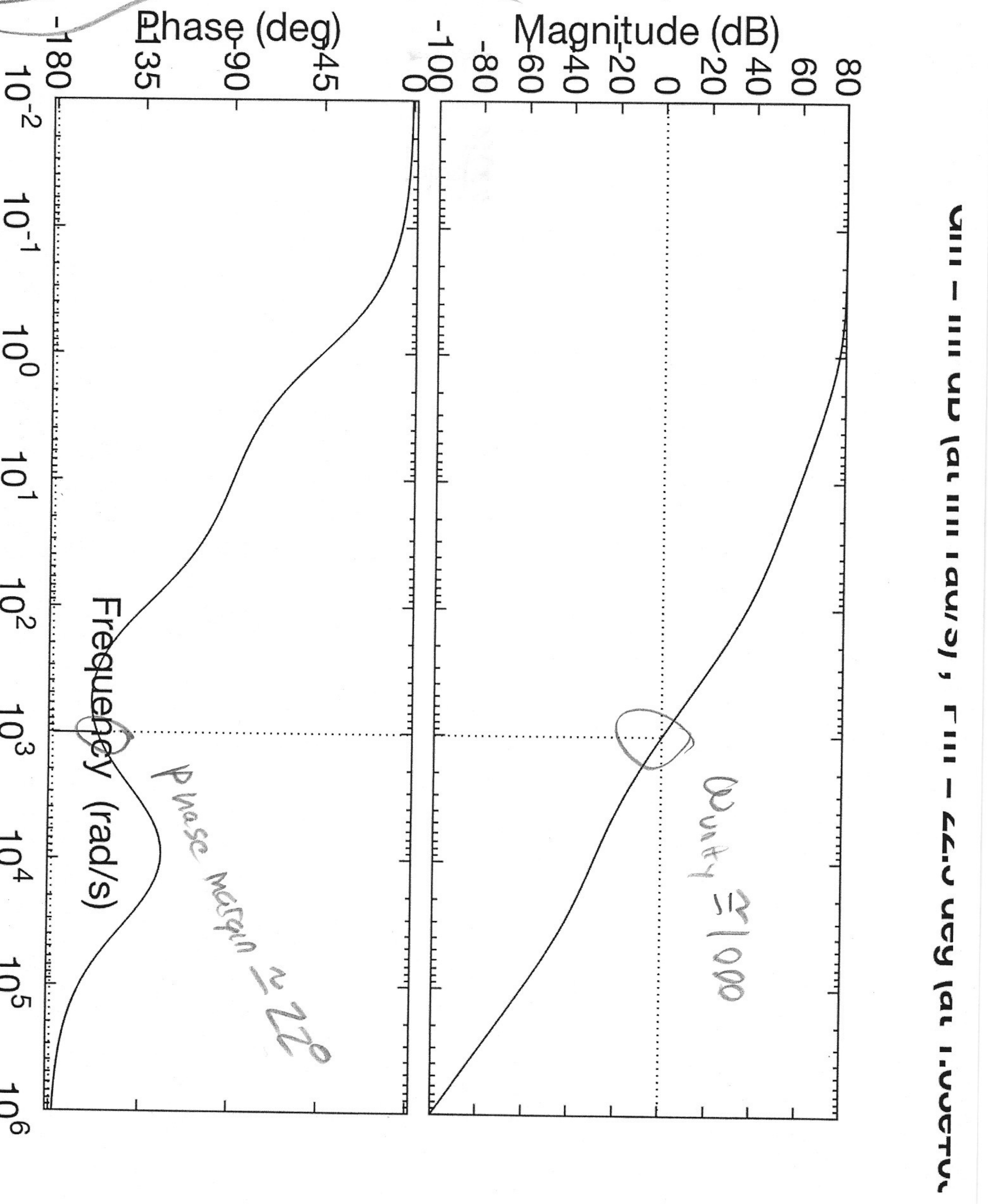


Actual Too Late $H_c(s) = 10\,\frac{s+10000}{s+100000}$ 13



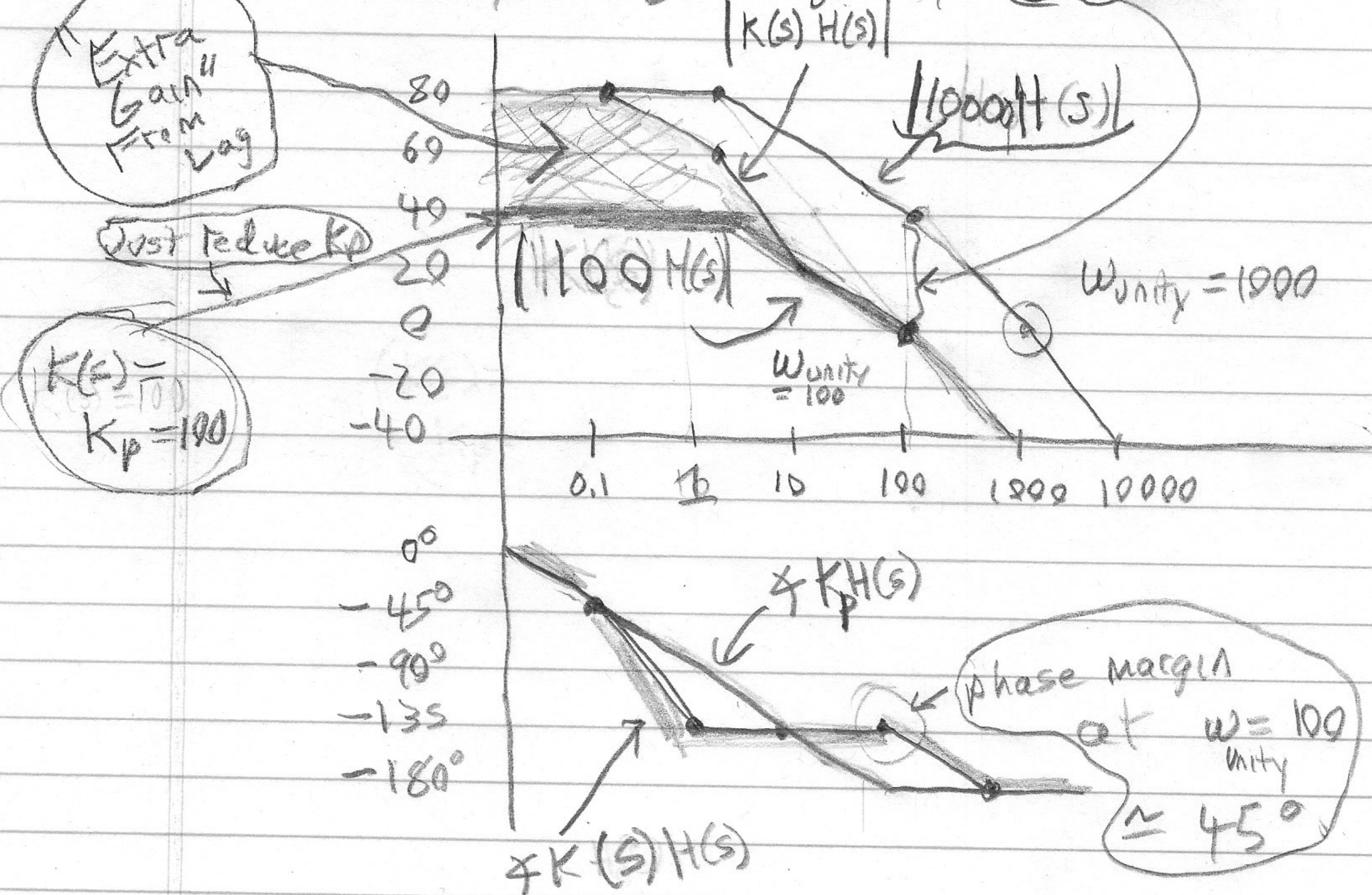
Lead a little late $\rightarrow H_c(s) = 10\left(\frac{s+3000}{s+30000}\right)$ (14)



2nd Idea

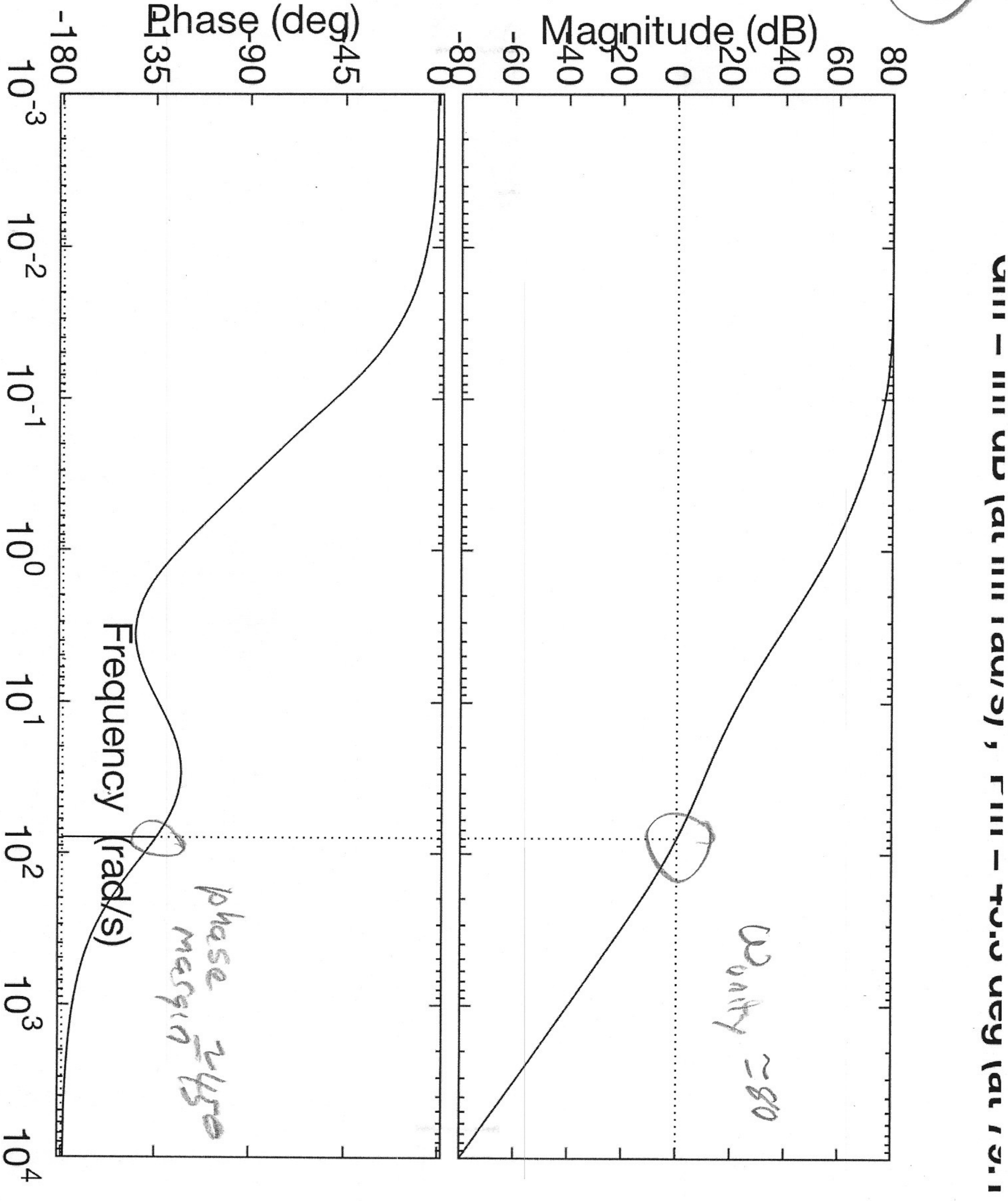
$$K(s) = 10000 \cdot \frac{1}{100}\left(\frac{s+10}{s+0.1}\right)$$

Lag to drop gain and reduce ω_{unity}. To reduce ω_{unity} to 100, reduce gain by 40dB!



$$K(s)H(s) = \underbrace{10000}_{K_p}\frac{1}{100}\left(\frac{s+10}{s+0.1}\right)\left(\frac{100}{(s+1)(s+100)}\right)$$

Actual Lag $K(s) = 10^4 \frac{1}{100}\left(\frac{s+10}{s+0.1}\right)$ (16)



Lag Too Soon $K(s) = \frac{10^4}{100}\left(\frac{s+1}{s+0.01}\right)$

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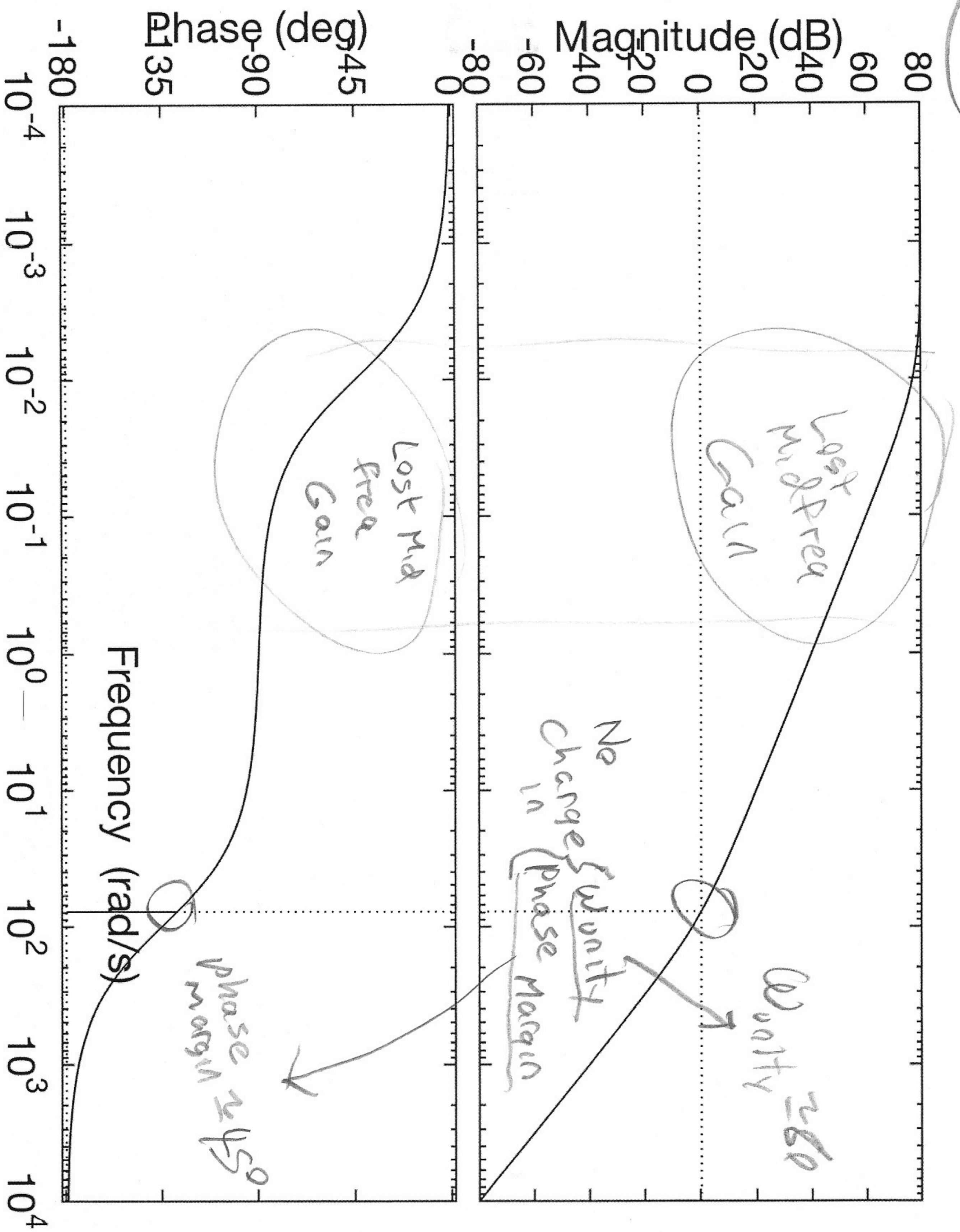

Lag Way Too Soon $K(s) = \frac{10^4}{100}\left(\frac{s+0.1}{s+0.001}\right)$

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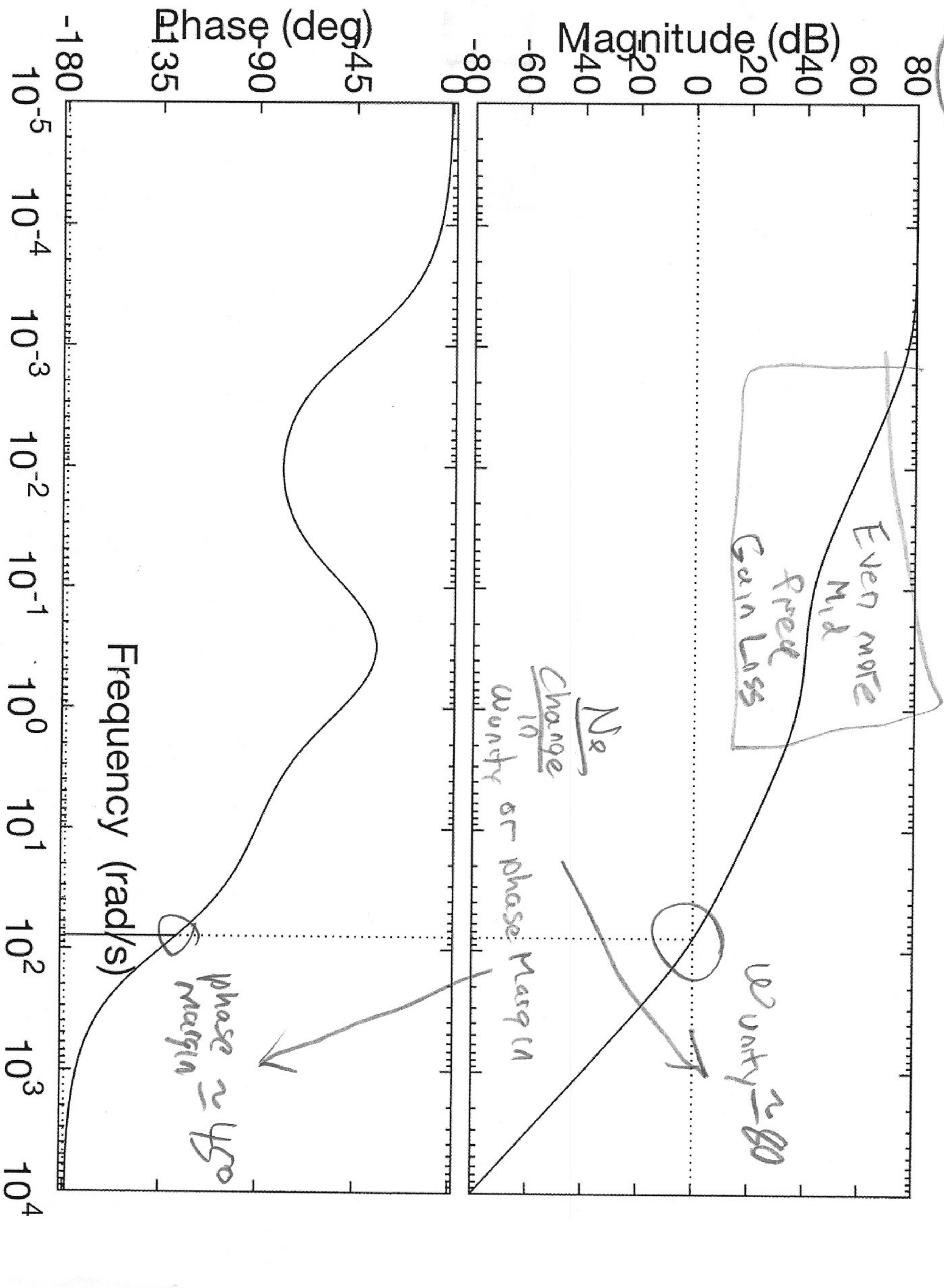


Observations

1) Lead compensators can be used to increase phase margin $\boxed{\text{without}}$ reducing gain at frequencies below ω_{unity}

$$|K H_{lead}(j\omega) H(j\omega)| \simeq |K H(j\omega)| \quad \omega < \omega_{unity}$$

2) Lead zero and pole must $\boxed{\text{surround}}$ ω_{unity} to be effective (Though compensated ω_{unity} may not equal uncompensated ω_{unity})

3) If the phase margin is acceptable at frequencies below ω_{unity}, $\boxed{\text{low-frequency Lag}}$ can be used to reduce the ω_{unity}.

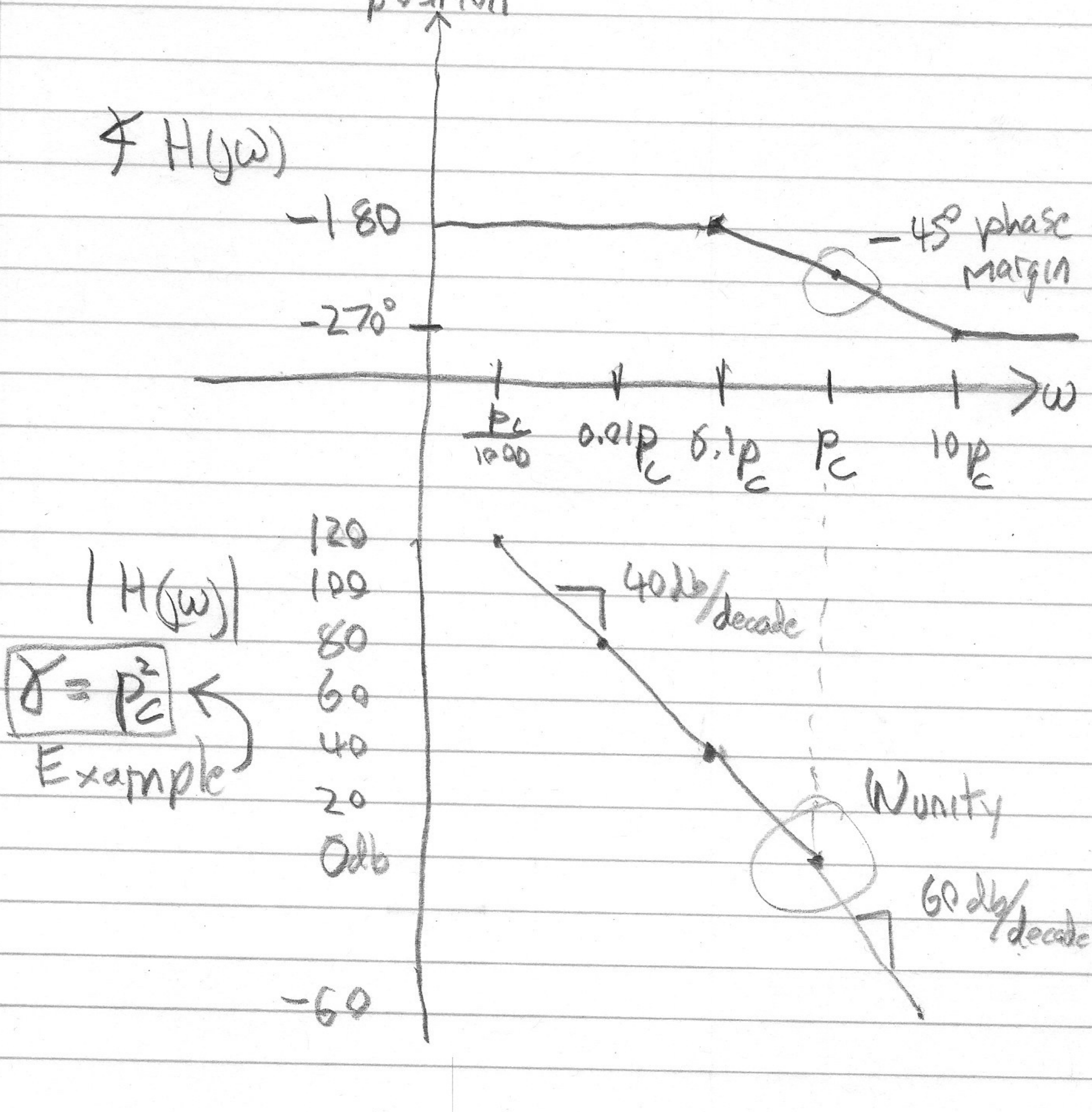
4) For many systems H, if a lag with pole p and zero q produces a given phase margin, then any $\alpha p, \alpha q$, $\alpha < 1$, will NOT reduce the phase margin.

Usual Mechanical Example

$$H(s) = \frac{\gamma(-p_c)}{s^2(s-p_c)}$$

Force to position

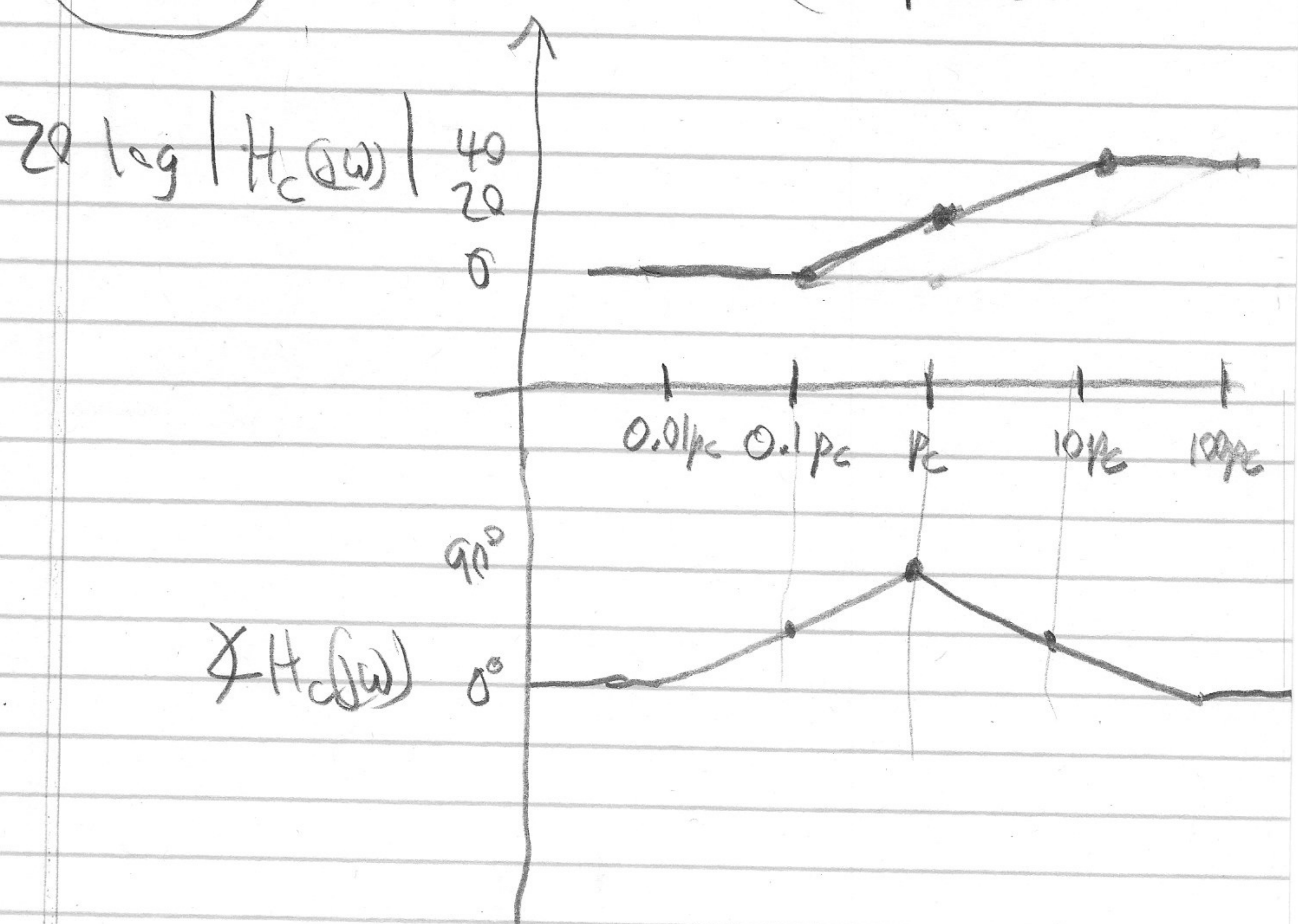
command → Force



1) <u>Lead Alone</u> Could work

pole-zero seperation $= \underline{\underline{100}}$!

$$H_c(s) = 100\left(\frac{s+p_c/10}{s+10p_c}\right)$$



Can you draw the diagrams for

$20\log|H_c(j\omega)H(j\omega)|$ and $\measuredangle H_c(j\omega)H(j\omega)$?

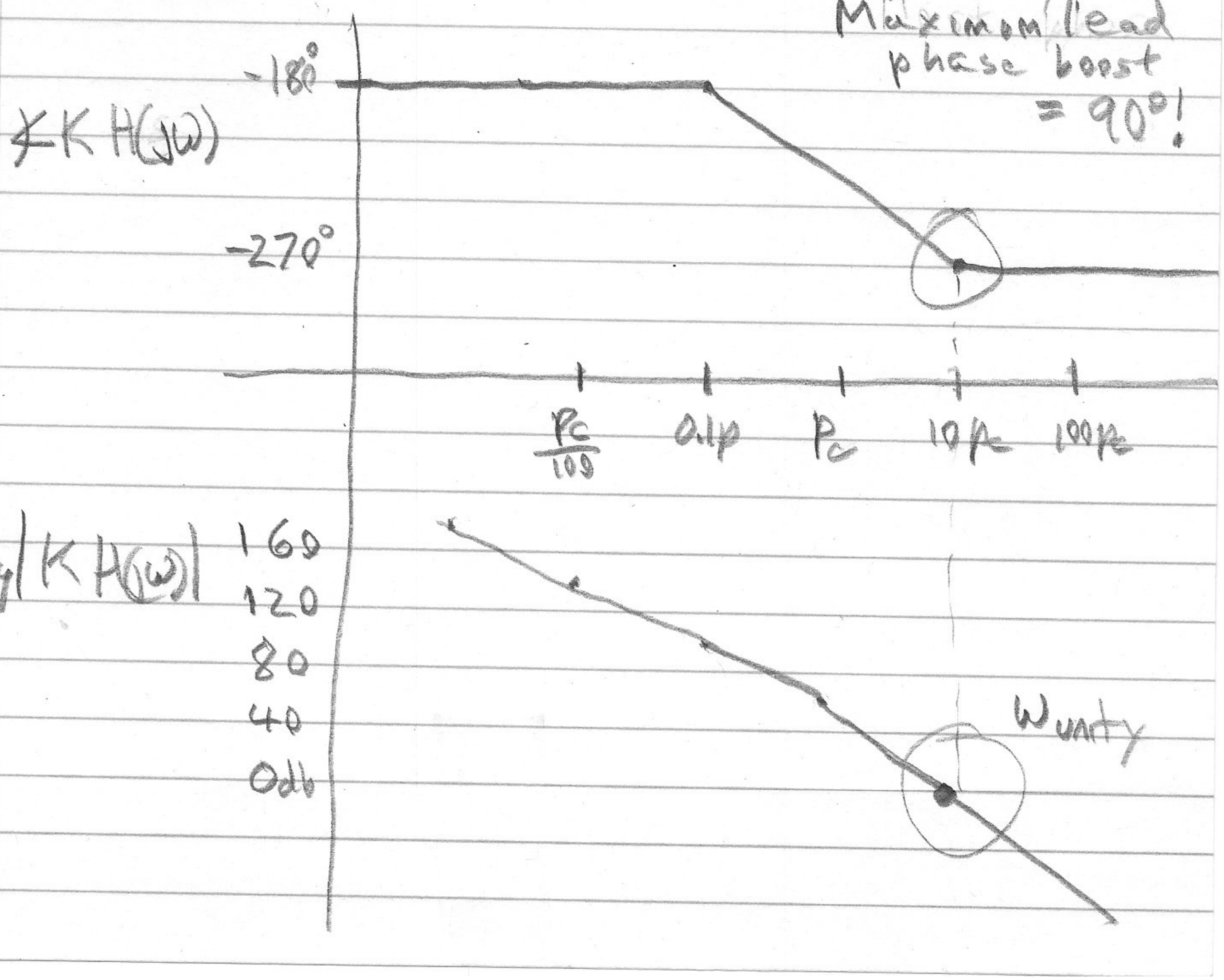
If $K_p = 1000$

with $\underbrace{K_p \cdot H_c(s)}_{K(s)} \cdot \dfrac{-p_c^3}{s^2(s-p_c)}$

Can $H_c(s)$ be a lead compensator and achieve a phase margin near $45°$?

Answer: No! Phase margin at $\omega_{unity} \approx -90°$

Maximum lead phase boost $= 90°$!



For K=1000 Case
Combine Lead and Lag

Drops the gain to make $\omega_{unity} \approx P_c$

$$H_{lag}(s) = \frac{1}{1000}\left(\frac{s - P_c/100}{s - P_c/100{,}000}\right)$$

"Bumps" the phase up at ω_{unity}

$$H_{lead}(s) = 100\left(\frac{s - P_c/10}{s - 10P_c}\right)$$

Can you plot the diagrams?