In these two lectures, My apologies for failing to communicate a point I think is important. Unlike looking at specific examples, using the linear algebra abstraction can help identify what is impossible. Though at this stage, it may not help identify how to accomplish what might be possible.

6.3100/2 2/24→26/25

Vector First order D.E.

usually 1 input

$$\underset{N}{x[n]} = \underset{N\times N}{A}\, x[n-1] + \underset{N\times \#in}{B}\, u[n-1]$$

Soln
$$x[n] = A^n x[0] + \sum_{m=0}^{n-1} A^{n-1-m} B u[m]$$

Steady State

Assume $u[n] = u_0 \ \forall n$

$$\lim_{n\to\infty} x[n] = x[\infty]$$

Does it Exist

What is it

If $x[\infty]$ Exists then

$\infty - 1 = \infty$

By assumption

$$x[\infty] = A x[\infty] + B u_0$$

$$x[\infty] - A x[\infty] = B u_0$$

$$\Downarrow$$

$$(I - A) x[\infty] = B u_0$$

$$x[\infty] = \underbrace{(I - A)^{-1} B}_{\text{hmm}} u_0$$

Exists?

← Later

Exists?

$$x[n] = A^n x[0] + \sum A^{n-1-m} B u_0$$

If A diagonizable $\quad A = V \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_N \end{bmatrix} V^{-1}$

$$A^n = V \lambda^n V^{-1}$$

$$V^{-1} x[n] = V^{-1} V \lambda^n V^{-1} x[0] + V^{-1} \sum V \lambda^{n-1-m} V^{-1} B u_0$$

$$\hat{x}[n] = \lambda^n \hat{x}[0] + \sum \lambda^{n-1-m} \hat{B} u_0$$

$$\tilde{x}_i[n] = \lambda_i^n \tilde{x}[0] + \sum \lambda_i^{n-1-m} \hat{B}_i u_0$$

If $|\lambda_i| < 1 \ \forall i \Rightarrow \hat{x}[\infty]$ Exists

$V \hat{x}[\infty] = x[\infty]$

Exists

Path Follow Robot Example

$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \underbrace{\overbrace{\begin{bmatrix} 1 & \Delta T S \\ -\Delta T K_p & 1-\Delta T K_\theta \end{bmatrix}}^{A}}\begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix}}^{B} \underset{\text{desired distance}}{d_d[n-1]}$$

$$\theta[n] = \theta[n-1] + \Delta T \underbrace{\left(K_p\left(d_d[n-1] - d[n-1]\right) + K_\theta\left(-\theta[n-1]\right)\right)}_{\omega[n-1]}$$

$\omega[n-1]$ ← angular velocity

Steady-State (if Exists)

Assume $d_d[n] = d_d[\infty]$ ← for all n

$$\begin{bmatrix} d[\infty] \\ \theta[\infty] \end{bmatrix} = \left[(I-A)^{-1} B\right] d_d[\infty]$$

$$(I-A)^{-1} = \left(\Delta T \begin{bmatrix} 0 & -S \\ K_p & K_\theta \end{bmatrix}\right)^{-1} = \frac{1}{\Delta T}\begin{bmatrix} \frac{K_\theta}{S K_p} & \frac{1}{K_p} \\ \frac{-1}{S} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \Delta T K_p \end{bmatrix}$$

$$(I-A)^{-1} B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} d[\infty] \\ \theta[\infty] \end{bmatrix} = \begin{bmatrix} d_d[\infty] \\ 0 \end{bmatrix}$$

Steady-State $\theta[\infty] = 0$ $d[\infty] = d_d[\infty]$

Steady State Exists?

Subtract
$$x[n] = A\,x[n-1] + B\,u[n]$$
$$-\quad x[\infty] = A\,x[\infty] + B\,u[\infty]$$

$= u[\infty]$ for steady state

Define $e[n] \equiv x[n] - x[\infty]$

$\lim_{n\to\infty} A^n = 0$ if eig(A) all <1 in mag.

$$e[n] = A\,e[n-1] = A^n e[0]$$

If $x[\infty]$ exists $\Longleftrightarrow$ $e[n] \to 0$

or $\underbrace{x[n] = A^n x[0] + \sum_{m=0}^{n-1} A^{n-1-m} B\,u[\infty]}_{\text{total solution}}$

If $A = V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix} V^{-1}$ (A is diagonalizable)

V: matrix of e vector

V^{-1}: inverse of matrix of e vectors

$$A^n = V \underbrace{\begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_N^n \end{bmatrix}}_{\lambda^n} V^{-1}$$

$$x[n] = V \lambda^n V^{-1} x[0] + \sum_{m=0}^{n-1} V \lambda^{n-1-m} V^{-1} B\,u[\infty]$$

$$\underbrace{V^{-1} x[n]} = \lambda^n \underbrace{V^{-1} x[0]} + \sum_{m=0}^{n-1} \lambda^{n-1-m} V^{-1} B\,u[\infty]$$

$$\tilde{x}[n] = \lambda^n \tilde{x}[0] + \sum \lambda^{n-1-m} \tilde{B}\,u[\infty]$$

or $\tilde{x}_i[n] = \lambda_i^n \tilde{x}_i[0] + \sum \lambda_i^{n-1-m} (\tilde{B}\,u[\infty])_i$

scalar case: $x_i[\infty]$ exists if $|\lambda_i| < 1$!

Back to the Robot

$$A = \begin{bmatrix} 1 & \Delta T S \\ -\Delta T K_p & 1-\Delta T K_\theta \end{bmatrix}$$

$$\text{eig}(A) \Rightarrow (\lambda - 1)(\lambda - (1-\Delta T K_\theta)) + (\Delta T)^2 S K_p = 0$$

$$\Rightarrow \lambda^2 - (2+\Delta T K_\theta)\lambda + 1 + \Delta T K_\theta + (\Delta T)^2 S K_p = 0$$

$$\lambda_1, \lambda_2 = 1 - \frac{\Delta T K_\theta}{2} \pm \Delta T \sqrt{\frac{K_\theta^2}{4} - K_p S}$$

$|\lambda_i| < 1$ if K_θ & K_p chosen well

Use Matlab script

Consider a Disturbance

$$x_{dist}[n] = A\,x_{dist}[n-1] + B\,u[n-1] + B_d\,u_d[n-1]$$

disturbed $X[n]$ disturbance

In steady-state

Assume $u[n] = u[\infty]$

$u_d[n] = u_d[\infty] \Leftarrow$ Normalize to 1

$$x_{dist}[n] = A\,x_{dist}[n-1] + B\,u[\infty] + B_d\,u_d[\infty]$$

Not disturbed steady state

$$X[\infty] = A\,x[\infty] + B\,u[\infty]$$

subtract

$$x_{dist}[n] - X[\infty] = A\left(x[n-1] - X[\infty]\right) + B_d\,u_d[\infty]$$

$$\hat{e}[n] \equiv X_{dist}[n] - X[\infty]$$

$=1$

$$\hat{e}[n] = A\,\hat{e}[n-1] + B_d\,u_d[\infty]$$

$$\hat{e}[\infty] = (I - A)^{-1} B_d$$

steady-state error due to disturbance

For Path Following Robot

$$(I - A)^{-1} = \frac{1}{\Delta T}\begin{bmatrix} \frac{K_\theta}{5K_p} & \frac{1}{K_p} \\ -\frac{1}{5} & 0 \end{bmatrix}$$

$$\hat{e}[\infty] = \underbrace{\frac{1}{\Delta T}\begin{bmatrix} \frac{K_\theta}{5K_p} & \frac{1}{K_p} \\ -\frac{1}{5} & 0 \end{bmatrix}}_{I - A} \begin{bmatrix} B_d[1] \\ B_d[2] \end{bmatrix} u_d[\infty]$$

disturbance in distance eqn

disturbance in θ eqn

Blowing Wind constant velocity V_w

$$d[n] = d[n-1] + \Delta T\, S\, \theta[n-1] + \Delta T \cdot \left(\text{Wind Vel}\right)$$

$$B_d = \begin{bmatrix} \Delta T \\ 0 \end{bmatrix} \left(\text{Wind Vel}\right) \leftarrow = 1 \text{ (Normalizing)}$$

$$(I - A)^{-1} B_d = \begin{bmatrix} \frac{K_\theta}{5K_p} \\ -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} d_{dist}[\infty] - d[\infty] \\ \theta_{dist}[\infty] - \theta[\infty] \end{bmatrix}$$

Steady State → distance error → 0 as $K_p \to \infty$ "high gain"

→ angle error $\not\to$ 0 as $K_p \to \infty$

increasing K_p does not affect Angle! error

Adding Integral (sum)

New state for path following robot

$$q[n] = q[n-1] - \Delta T\left(d_d[n] - d[n]\right)$$

along with

$$d[n] = d[n-1] + \Delta T\, S\, \theta[n-1]$$

$$\theta[n] = \theta[n-1] + \Delta T\, \omega[n-1]$$

we control

$$\omega[n] = K_p\left(d_d[n] - d[n]\right) + K_\theta\left(-\theta[n]\right) + K_i\left(-q[n]\right)$$

for integral even though q is a sum

Constant normalized disturbance

$$\begin{bmatrix} d[n] \\ \theta[n] \\ q[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta T S & 0 \\ -\Delta T K_p & 1-\Delta T K_\theta & -\Delta T K_i \\ \Delta T & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} d[n-1] \\ \theta[n-1] \\ q[n-1] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \Delta T K_p \\ -\Delta T \end{bmatrix}}_{B} d_d[n] + \underbrace{\begin{bmatrix} B_{d_1} \\ B_{d_2} \\ B_{d_3} \end{bmatrix}}_{B_d} \cdot 1$$

Signs could all be flipped if q defined differently

Steady State disturbance

$$\hat{e}[\infty] = (I - A)^{-1} B_d$$

If eig(A) all are less than one in magnitude

for what values of K_p, K_D, K_i are evals of

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \Delta T \begin{bmatrix} 0 & S & 0 \\ -K_p & -K_D & -K_i \\ 1 & 0 & 0 \end{bmatrix} \quad (I-A) =$$

Inside unit circle?

Reorganizing

$$\text{evals}(A) = 1 - \Delta T \left(\text{evals} \begin{bmatrix} 0 & -S & 0 \\ K_p & K_D & K_i \\ -1 & 0 & 0 \end{bmatrix} \right)$$

Use Spectral Mapping Theorem

What circle should these evals be in?

Does this help? Probably not use the computer!

What about Steady-State

$$\hat{e}[\infty] = \left(\Delta T \begin{bmatrix} 0 & 5 & 0 \\ -K_p & -K_d & -K_i \\ 1 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B_d(1) \\ B_d(2) \\ B_d(3) \end{bmatrix}$$

disturbance from steady state

Want $\hat{e}[\infty] = 0$

Steady-state insensitive to disturbance!

Is this possible?

<u>If</u> $\hat{e}[\infty] = (I - A)^{-1} B_d \underbrace{u_d[\infty]}_{= 1 \text{ by normalization}}$

$\Rightarrow$ Assumes evals(A) <u>inside</u> unit circle

<u>Questions</u> Is there an A with evals inside U.C.

<u>such that</u>
(1) $(I - A)^{-1} B_d = 0$ for all B_d's
(2) $(I - A)^{-1} B_d = 0$ for any B_d?
(3) $((I - A)^{-1} B_d)_1 = \hat{e}_1[\infty] = 0$ for all B_d?

i.e. Can you design a stable system with any of these properties

<u>No!</u>

<u>(1)(2)(3)</u> Impossible!

Back to integral for Robot

$$\hat{e}[\infty] = (I - A)^{-1} B_d \, u_d[\infty]$$

=1 (Normalizing)

steady-state error due to disturbance

$$(I-A)^{-1} B_d = \left(\Delta T \begin{bmatrix} 0 & 5 & 0 \\ -k_p & -k_d & -k_i \\ 1 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} Bd_1 \\ Bd_2 \\ Bd_3 \end{bmatrix}$$

Nonzeros

$$\begin{pmatrix} 0 & 0 & \xi \\ \xi & 0 & 0 \\ \xi & \xi & \xi \end{pmatrix}$$

Bd_3 is always zero (the ϕ equation is part of your controller)

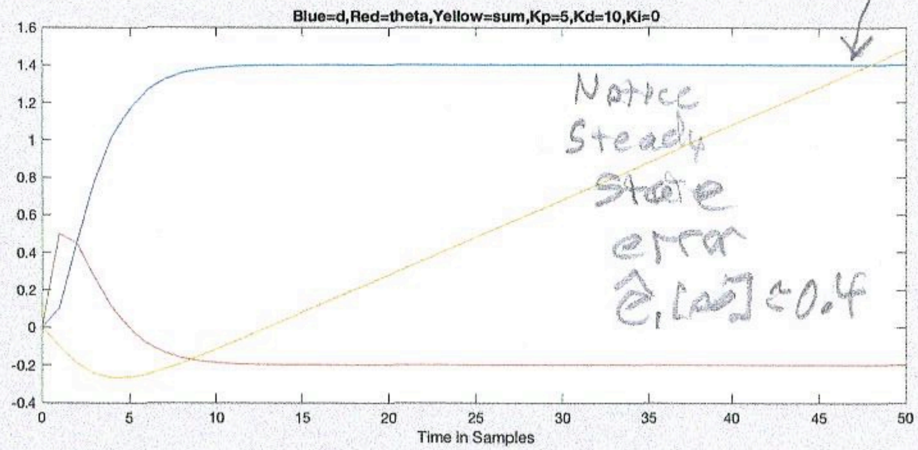
Bd_1 is a disturbance in distance equation

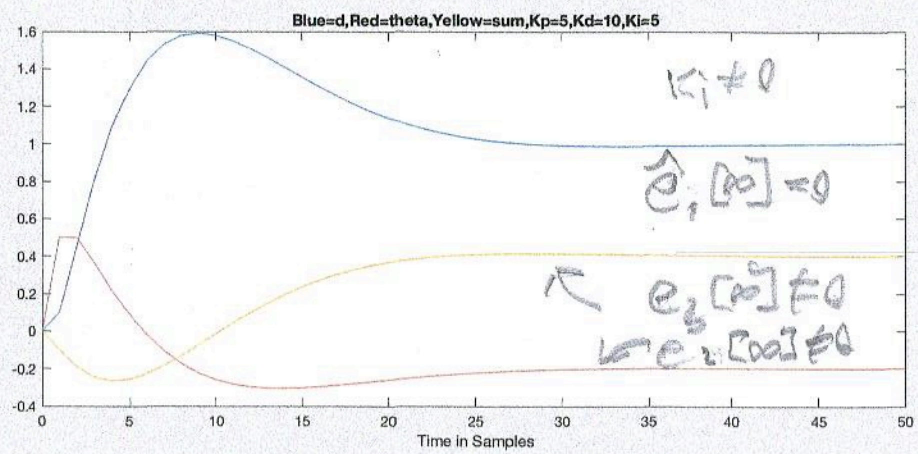
Bd_2 is a disturbance in angle eqn

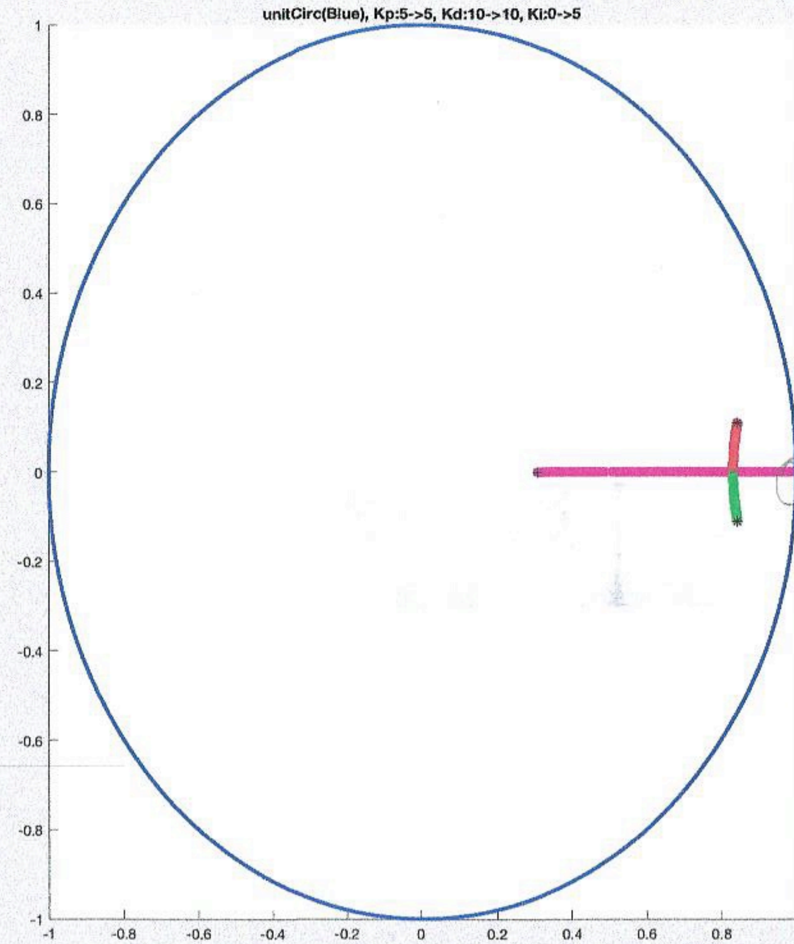
So with integral controller

$\hat{e}_1[\infty] = 0$ for any Bd_1 or Bd_2 $\hat{e}_3[\infty] \neq 0$!

$B_d = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}$







unstable integrator

$$B_d = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}$$