MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.3100: Intro to Modeling and Control—Spring 2025

Code of Arms - Due 3pm, 03/10/25

A Mystery From Lab

In the last part of the code-of-arms lab, we asked you to find values for the controller gains K_p and K_d that stabilized the step responses from a controller with $K_i = 10$. As most of you discovered, the step responses either settled very slowly or exhibited substantial overshoot, like the arm angle behavior plotted in the top half of the picture below (measured angle in blue, desired angle in red). In the bottom half of the picture below, we show a plot of the arm behavior (with the same controller gains) after we changed the Teensy sketch slightly. As you can see in the figure, when we deleted the $K_p\theta_d[n]$ term from the formula for the motor motor command. In this problem set we are going to examine why.

We have been describing discrete-time systems with inputs and disturbances as

$$x[n] = Ax[n-1] + Bu[n-1] + B_d u_d[n-1]$$

where $x[n]$ is an N-length vector of states, A is an $N \times N$ matrix whose eigenvalues are the system's natural frequencies, the input and disturbance matrices B and B_d are $N \times \#inputs$ and $N \times \#disturbers$, though we often consider systems with a scalar input (e.g. $u[n] = \theta_d[n]$ in the propeller arm case) and a scalar disturbance, in which case B and B_d are $N \times 1$ matrices.

For most of the experiments in the code-of-arms lab, we were measuring zero input responses. That is, we lifted the arm to set a non-zero initial angle, but zero initial angular velocity and acceleration, and then dropped the arm and monitored its behavior. Since the inputs and disturbances were zero for these experiments, we could ignore B and B_d, and focus on selecting controller gains for which the natural frequencies (aka the eigenvalues of A) were all less than one in magnitude. We can not ignore B and B_d when analyzing the last experiment of the lab, where we used sum (integral) feedback to improve input tracking and disturbance rejection. For that last experiment, we were NOT measuring zero input responses, we tested the arm with input steps instead of lifting-and-letting-go. And in addition, we dropped Lego U's on the arm to disturb it. So, we need to look beyond the eigencondition on A, and examine the impact of B and B_d.

In this postlab, we will focus on analyzing steady-state behavior, which means we are making several assumptions. First, we are assuming that for the selected controller gains, the eigenvalues of A are all less than one in magnitude, so there is a steady state and $(I - A)^{-1}$ exists. Second, we will assume the input and the disturbance are constants (which we will normalize for convenience as $u[n] = 1$ and $u_d[n] = 1$ for all n). Experimentally that means we change the input or the disturbance once, at the start of an experiment, but then hold them fixed (well not quite, we just hold the input fixed long enough for the arm's behavior to settle).

If A satisfies the eigencondition (its eigenvalues are less than one in magnitude) the disturbance-free steady-state $x[\infty]$ is then given by

$$x[\infty] = (I - A)^{-1} B.$$

and the difference between the steady-state with and without disturbance is given by

$$x_{dist}[\infty] - x[\infty] = (I - A)^{-1} B_d.$$

We can often arrange for the $N \times N$ matrix $I - A$ to have structural properties which guarantee zeros in the $N \times N$ matrix $(I - A)^{-1}$, *independent of model parameters or gains.* Such structural zeros are important! If the i, j entry of $(I - A)^{-1}$ is always zero, regardless of controller gain (assuming stability), then a non-zero in the j^{th} row of B or B_d has no effect on $x_i[\infty]$. As we will see below, the structure of the matrices describing our propeller arm system lead to many structural zeros in $(I - A)^{-1}$ and provide us with numerous gain-independent insights.

This problem set has only one problem, but many, many parts.

A,Matrix Structure: Suppose each of the first L rows of an invertible $N \times N$ matrix M has only one nonzero. That is, $M_{i,i+1}$ is the only nonzero in row i, for $i \in 1, ..., L$. As an example, suppose $L = 3$ and $N = 5$, then

$$M = \begin{bmatrix} 0 & M_{1,2} & 0 & 0 & 0 \\ 0 & 0 & M_{2,3} & 0 & 0 \\ 0 & 0 & 0 & M_{3,4} & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}.$$

where the $*$ is used to denote a matrix entries that are possibly non-zero.

If M is invertible, then M^{-1} exists and each of its rows 2 through $l+1$ has exactly one nonzero. That is, $M^{-1}_{i+1,i}$ is the only nonzero in row $i+1$, for $i \in 1, ..., L$. For the $L = 3$ and $N = 5$ example above,

$$M^{-1} = \begin{bmatrix} * & * & * & * & * \\ (M^{-1})_{2,1} & 0 & 0 & 0 & 0 \\ 0 & (M^{-1})_{3,2} & 0 & 0 & 0 \\ 0 & 0 & (M^{-1})_{4,3} & 0 & 0 \\ * & * & * & * & * \end{bmatrix}.$$

Note that these are structural properties, and do NOT depend on specific values for the matrix entries!

What are the L values $(M^{-1})_{i+1,i}$, $i \in 1, ..., L$, as a function of $M_{i,i+1}$, $i \in 1, ..., L$ (hint, use the fact that $MM^{-1} = I$)?

B,Reorder For Structure: For checkoff five of the code-of-arm lab, we calibrated a 4×4 A matrix that models the propeller arm system with proportional and derivative (PD) feedback control. In checkoff six, we extended that model to include an integral (or sum) term. Then we generated a 5×5 A matrix that models the propeller arm system with proportional, integral, and derivative (PID) feedback control.

Please show a re-ordering of the states ($\theta_a[n]$, $\theta_a[n-1]$, $\omega_a[n]$, $\alpha_a[n]$ and $sum[n]$), *and the associated A matrix* that results in an $I - A$ matrix with the structural property described in the first part of this problem. For the PD case, find an ordering for which the structural property holds with $L = 2$, and for the PID case, for $L = 3$. Please help make this problem easier to grade, and for both cases, order the $\theta_a[n-1]$ state last (we do NOT mean the $\theta_a[n]$! state).

C, Inverses: Using your state orderings, determine $(I - A)^{-1}$ for both the PD and PID cases. Please give your answers in the same format as the 5×5 M^{-1} example above. That is, determine formulas for the values in the middle L rows, but use asterisks to represent any potentially non-zero entries in the first and last rows.

D,Killer B's: Consider the simplified version of the motor command (in which we assume $m = 1$ and $\theta_d[n] - \theta_d[n-1] \approx 0$),

$$c[n] = K_p\theta_d[n] + K_p(-\theta_a[n]) + K_d\left(-\frac{\theta_a[n] - \theta_a[n-1]}{\Delta T}\right) + K_i sum[n].$$

For the simplified motor command above, and your state reorderings, what are the B matrices for the PD and PID cases? And how do your B matrices change if we eliminate the $K_p\theta_d[n]$ term from $c[n]$ (just like in the modification of the Teensy sketch).

Don't forget! The difference equation for $sum[n]$ depends on $\theta_d[n]$, and therefore contributes to B,

$$sum[n] = sum[n-1] + \Delta T \left(\theta_d[n-1] - \theta_a[n-1]\right).$$

E,Get it Together: For the PD and PID cases, the non-zero patterns of their respective $(I-A)^{-1}$ and B matrices tell you which steady-states (of θ_a, ω_a, α_a and sum) could be effected by discarding the $K_p\theta_d[n]$ term from the motor command. For the PD case and PID cases, which steady-states could possibly be effected by dropping $K_p\theta_d[n]$ term from the motor command? How does this help explain part of the mystery described at the beginning of this problem set?

F, Disturber: Consider two possible disturbances: first, a drop in voltage for propeller motor power supply, and second, dropping a Lego U on to the propeller arm. For the PD and PID cases, which entries of the associated B_d matrix are nonzero for each of these two disturbances?

BEWARE: In our model for the propeller arm, we introduced a state variable that we deceptively referred to as arm angular acceleration. Despite its label, our state α_a is only part of the arm acceleration, the part due to thrust generated by the propeller motor. Note that *the total arm angular acceleration must be zero in steady-state*, but $\alpha_a[\infty]$ might not be zero, as it may be balancing contributions to arm acceleration from other forces (such as the gravity force due to a weight dropped onto the arm).

G, Disturbed: For the PD and the PID cases, and the two different disturbances, use the nonzero patterns of the asssociated B_d and $(I-A)^{-1}$ matrices to determine which steady-states (out of θ_a, ω_a, α_a and sum) could be effected be each disturbance. When using the PID controller, is there any physical disturbance that will effect the arm angle's steady state? Why would one describe a "disturbance" in the the equation for $sum[n]$ as "sensor noise"?