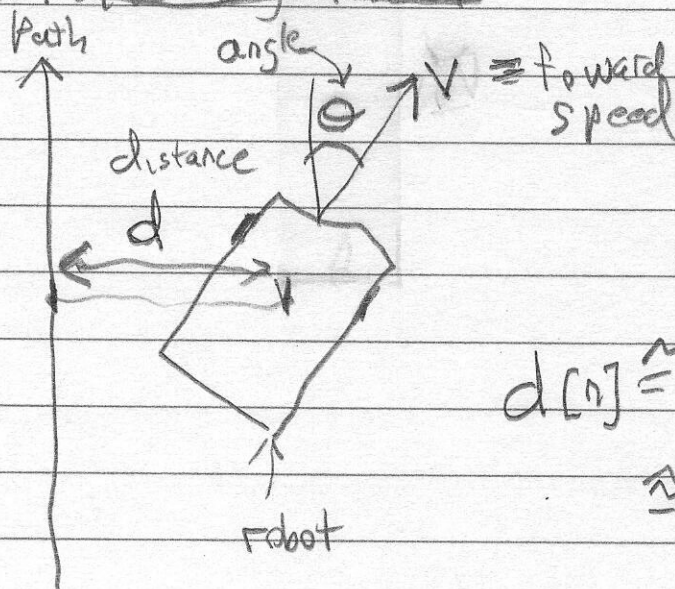


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(1)

Path-following Robot



$\Delta T = \text{sample period}$

$$d[n] \approx d[n-1] + \Delta T V \sin(\theta[n-1])$$

$$\approx d[n-1] + \Delta T V \theta[n-1]$$

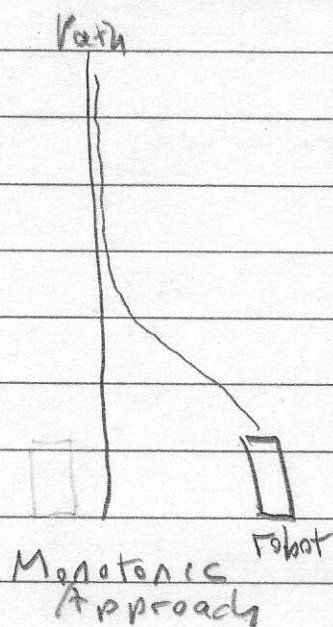
if θ is small
(and in radians!)

$\omega = \text{robot rotational speed}$
(we control)

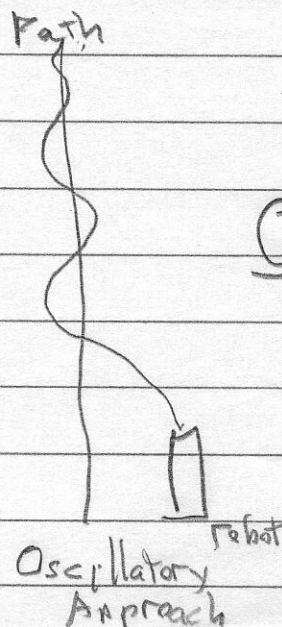
$$\theta[n] = \theta[n-1] + \Delta T \omega[n-1]$$

Suppose $\omega[n] = K_p (-d[n])$
(further from the path, faster we rotate)

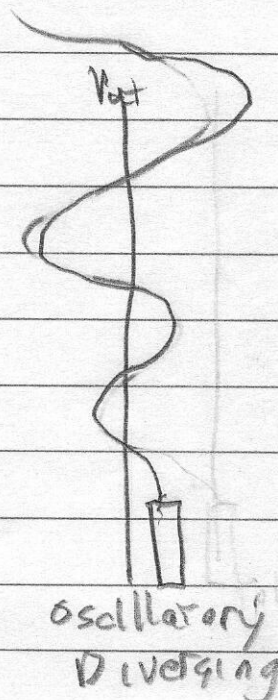
Possible Trajectories



OR



OR



Matrix Form

(2)

$$\begin{bmatrix} d[n] \\ e[n] \end{bmatrix} = \begin{bmatrix} 1 & \Delta T V \\ -\Delta T K_p & 1 \end{bmatrix} \begin{bmatrix} d[n-1] \\ e[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta T \end{bmatrix} u[n-1]$$

$$X[n] = A X[n-1] + B u[n-1]$$

ZIR $X[n] = A^n X[0]$

or $A V = V [\Lambda] \leftarrow \text{diagonalizable}$

then $\begin{bmatrix} d[n] \\ e[n] \end{bmatrix} = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \begin{bmatrix} V^{-1} \end{bmatrix} \begin{bmatrix} d[0] \\ e[0] \end{bmatrix}$

or $d[n] = C_1 \lambda_1^n + C_2 \lambda_2^n$

$$d[0] = C_1 \lambda_1^{\overset{=1}{0}} + C_2 \lambda_2^{\overset{=1}{0}}$$

$$d[1] = C_1 \lambda_1 + C_2 \lambda_2$$

$$\begin{bmatrix} d[1] \\ e[1] \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} d[0] \\ e[0] \end{bmatrix} \leftarrow \text{easy}$$

Does $d[n] \xrightarrow[n \rightarrow \infty]{} 0$? Yes if $|\lambda_1| < 1 \text{ \& } |\lambda_2| < 1$

Find λ 's

(3)

$\det(\lambda I - A) = 0$ λ is an eval
 2×2

$$\det \begin{pmatrix} \lambda - 1 & -\Delta T V \\ +\Delta T K_p & \lambda - 1 \end{pmatrix} = 0$$

$$\begin{aligned} (\lambda - 1)(\lambda - 1) + (\Delta T)^2 K_p V &= 0 \\ = \lambda^2 - 2\lambda + (1 + (\Delta T)^2 K_p V) &= 0 \end{aligned}$$

$$\lambda_1, \lambda_2 = 1 \pm \sqrt{1 - (1 + (\Delta T)^2 K_p V)}$$

$$= 1 \pm \sqrt{-(\Delta T)^2 K_p V}$$



$$= 1 \pm j \Delta T \sqrt{K_p V}$$



Imaginary

So

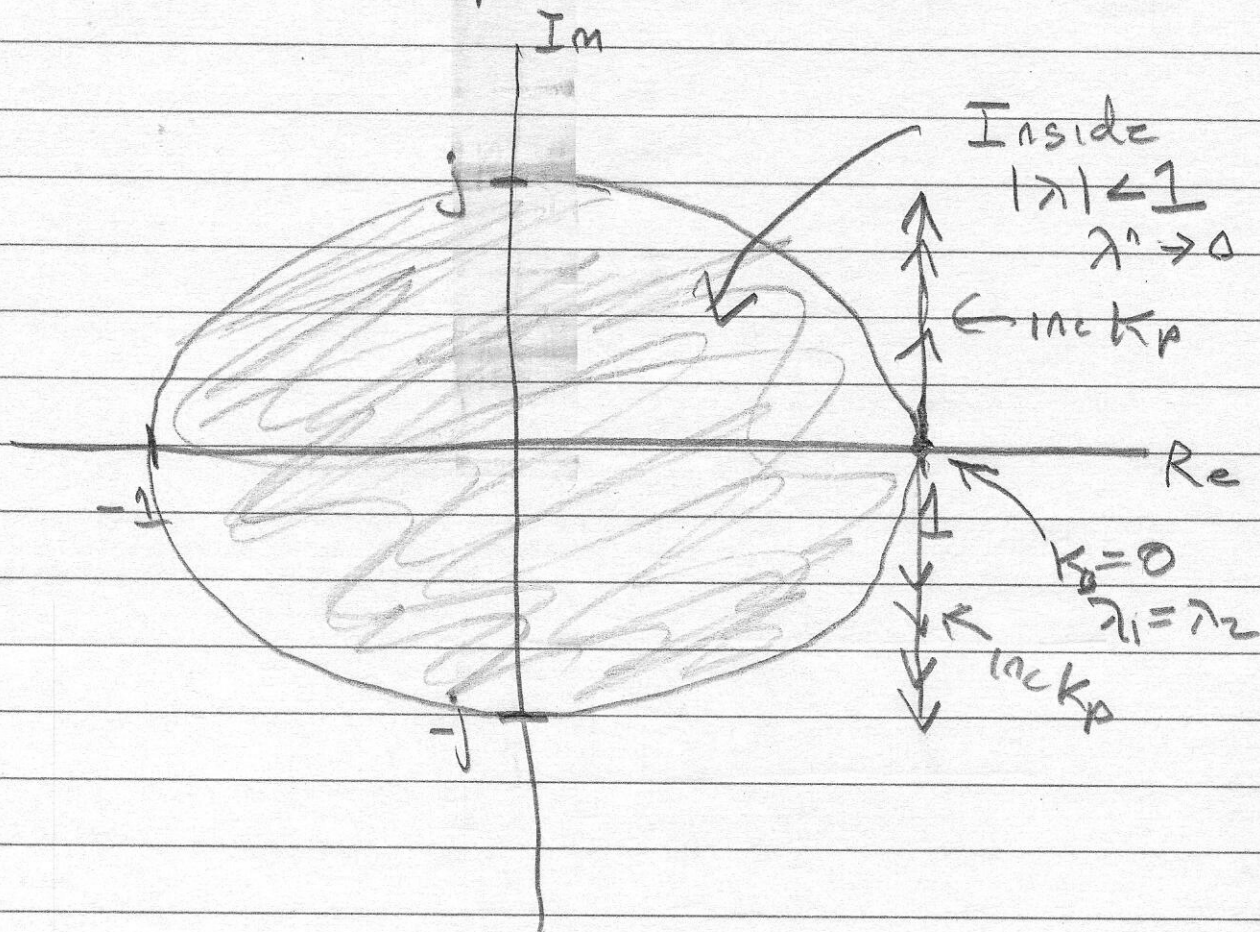
$$d[n] = C_1 \lambda_1^n + C_2 \lambda_2^n$$

$$= C_1 (1 + j \Delta T \sqrt{K_p V})^n + C_2 (1 - j \Delta T \sqrt{K_p V})^n$$

$$d[n] \rightarrow 0?$$

The Complex Plane

4



$$\lambda_1, \lambda_2 = 1 \pm j \Delta T \sqrt{K_p V}$$

So $|d[n]| \rightarrow \infty$ as $n \rightarrow \infty$
for all $K_p > 0$!

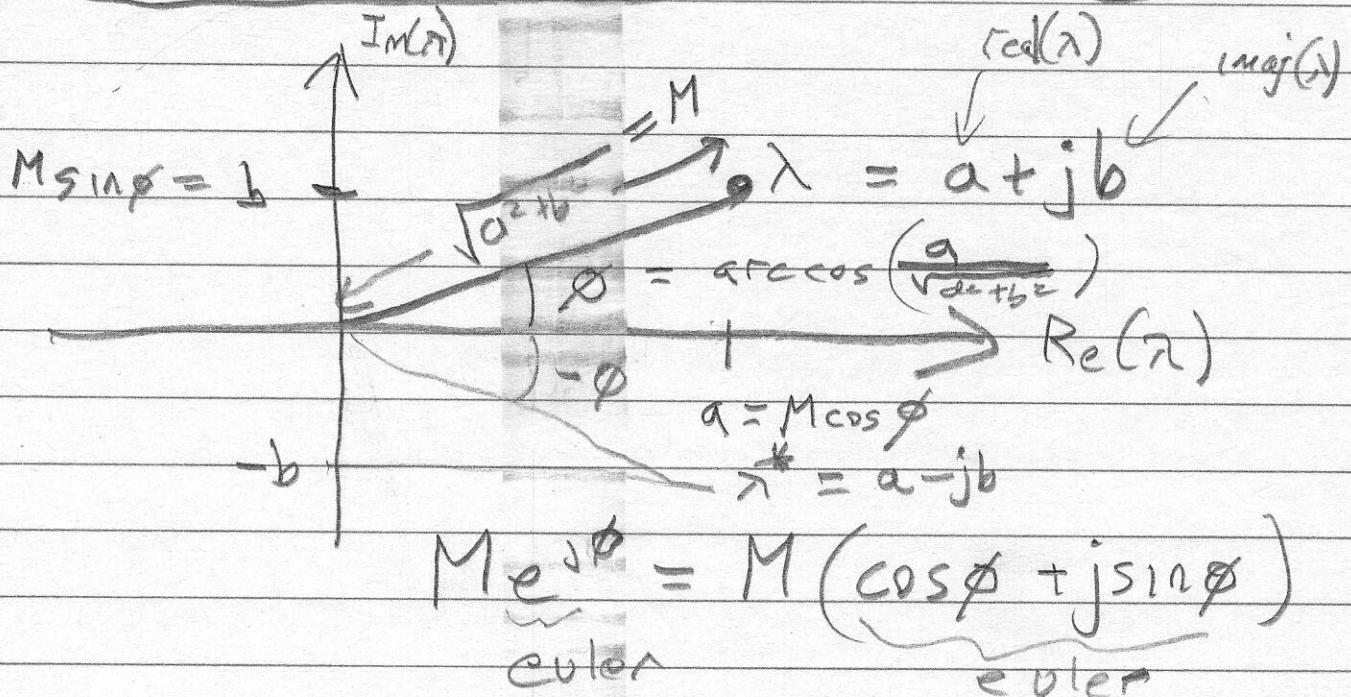
But why does it oscillate?
 And is

$$d[n] = C_1 (1 + j \Delta T \sqrt{K_p V}) + C_2 (1 - j \Delta T \sqrt{K_p V})$$

always real?
 real because $C_1 = C_2^*$ (complex conj)

Polar Form

(5)



$$\lambda_1^n = (M e^{j\phi})^n = M^n e^{jn\phi} = M^n (\cos n\phi + j \sin n\phi)$$

$$\lambda_2^n = (\lambda_1^*)^n = (M e^{-j\phi})^n = M^n e^{-jn\phi} = M^n (\cos n\phi - j \sin n\phi)$$

$$d[n] = C_1 M^n e^{jn\phi} + C_2 M^n e^{-jn\phi}$$

$$C_1 = M_c e^{j\phi_c} \quad C_2^* = C_1 = M_c e^{-j\phi_c}$$

$$d[n] = M_c M^n e^{j(n\phi + \phi_c)} + M_c M^n e^{-j(n\phi + \phi_c)}$$

$$= M_c M^n \cos(n\phi + \phi_c)$$

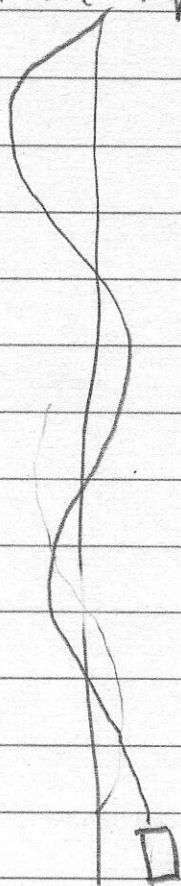
BTW $n_{\text{period}} \phi = 2\pi$ (one period)

$$n_{\text{period}} = \frac{2\pi}{\phi}$$

Robot Trajectories

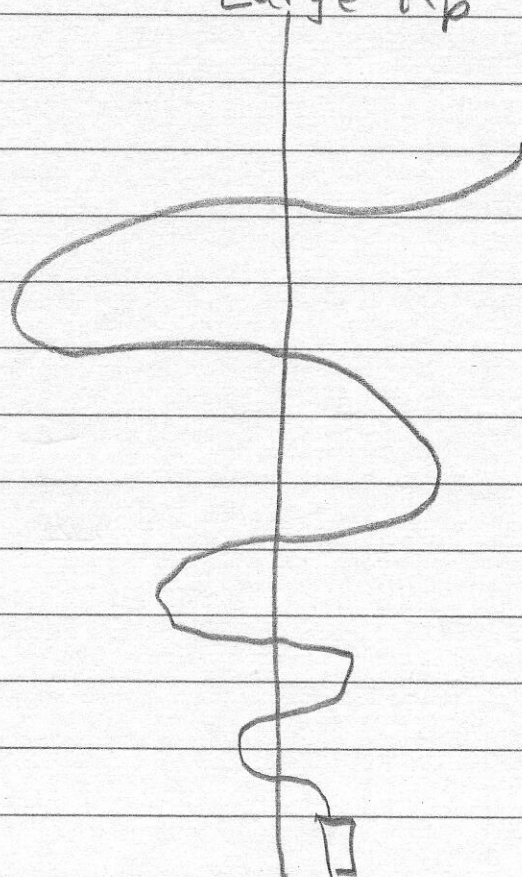
5A

small K_p



$|\lambda|$ slightly > 1
 imaginary part is small
 ϕ is small
 $\frac{2\pi}{\phi}$ large

Large K_p



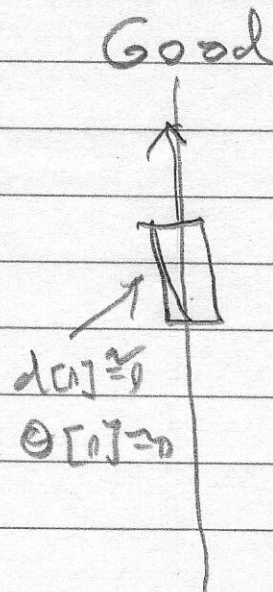
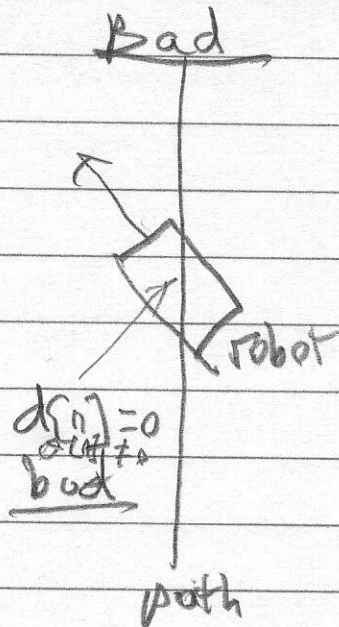
$|\lambda|$ much larger than 1
 imaginary part is large
 ϕ is large
 $\frac{2\pi}{\phi}$ is small

Suppose

⑥

$$u[n] = K_p(-d[n]) + K_o(-\theta[n])$$

Drive distance and angle to zero
simultaneously



$$\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix} = \begin{bmatrix} 1 & \Delta T V \\ -\Delta T K_p & 1 - \Delta T K_o \end{bmatrix} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix}$$

$$\lambda^2 - (2 + \Delta T K_o)\lambda + 1 + \Delta T K_o + \Delta T^2 V K_p = 0$$

$$\lambda_1, \lambda_2 = 1 - \frac{\Delta T K_o}{2} \pm \Delta T \sqrt{\frac{K_o^2}{4} - K_p V}$$

Need Plots of λ_1, λ_2 versus K_o, K_p