These notes have "Asides" on several pages (look for the thick black line). They cover background that you have likely seen before, or derivations you might not have seen.

First-Order Scalar

DT

$$y[n] = \lambda y[n-1] + \gamma u[n-1]$$

Gen Sol

$$y[n] = \lambda^n y[0] + \sum_{m=0}^{n-1} \lambda^{n-m-1} u[m]$$

(Proof by induction)

CT

$$\frac{d}{dt} y(t) = \lambda y(t) + \gamma u(t)$$

Gen Sol

$$y(t) = e^{\lambda t} y(0) + \int_0^t e^{\lambda(t-\tau)} u(\tau)\, d\tau$$

(Proof uses integration by parts)

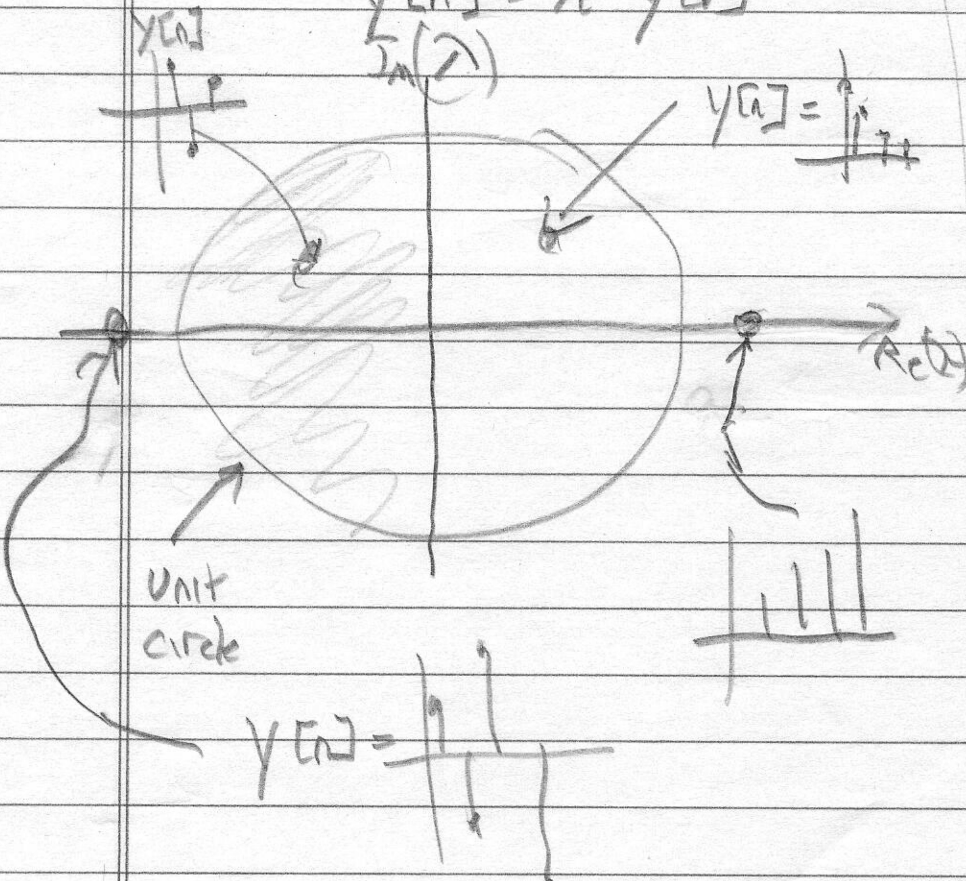
Same Linearity properties

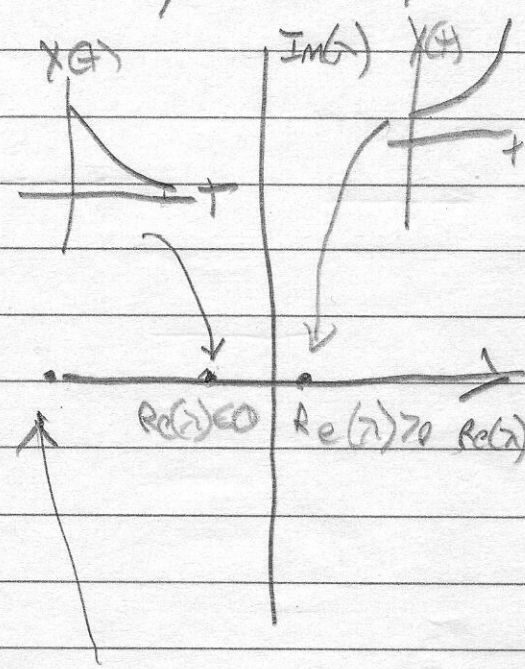
Z.I.R. $y[n] = \lambda y[n-1]$

$$y[n] = \lambda^n y[0]$$

Z.I.R. $\dot{y}(t) = \lambda y(t)$

$$y(t) = e^{\lambda t} y(0)$$





D.T. Stability $|\lambda| < 1$

C.T. Stability $Re(\lambda) < 0$

Frequency Response Calculations

2

<u>Given:</u> $\frac{d}{dt} y(t) = \lambda y(t) + \gamma u(t)$

$$u(t) = A\cos(\omega t + \phi) \leftarrow$$ sinusoidal input

Amplitude A

$$\underset{\substack{\uparrow \\ \text{cycles/sec}}}{2\pi f} = \underset{\substack{\uparrow \\ \text{radians/sec}}}{\omega}$$ $\rightarrow$ freq ω

angle ϕ

Simpler to analyze using eigenfunctions

<u>Use</u> $u(t) = \underset{\substack{\uparrow \\ \text{complex}}}{\underline{U}} \underbrace{e^{j\omega t}}_{\text{complex}}$

<u>Because</u> $\frac{d}{dt} u(t) = \frac{d}{dt} \underline{U} e^{j\omega t} = j\omega \underline{U} e^{j\omega t}$

Derivatives $\Rightarrow$ multiplication by scalar

<u>Aside</u> For linear operators $F(\,)$ on functions

e.g. $\underset{\substack{\uparrow \\ \text{linear operator}}}{\frac{d}{dt}} \underbrace{(y(t))}_{\text{function}}$

<u>If</u> $\underset{\substack{\uparrow \\ \text{oper}}}{F}(\underset{\substack{\uparrow \\ \text{function}}}{y(t)}) = \underset{\substack{\uparrow \\ \text{scalar}}}{\lambda} y(t)$ $\quad y(t) \equiv$ eigenfunction

$\lambda \equiv$ eigenvalue

<u>Like in Linear Alg eq.</u> $A\vec{x} = \lambda \vec{x}$ $\quad \vec{x} =$ eigenvector

$\lambda =$ eigenval

<u>So</u> for any $s \in \underset{\text{complex numbers}}{\mathbb{C}}$ e^{st} is efunc of $\frac{d}{dt}$

$$\frac{d}{dt} \overbrace{e^{st}}^{\text{eigenfunc}} = \overset{\text{eval}}{s}\, \underbrace{e^{st}}_{\text{eigenfunct}}$$

What is $\underline{U} e^{j\omega t}$

↑ complex scalar

Euler

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Aside

Proof: Generically $e^{j\phi} = \cos\phi + j\sin\phi$

Match Taylor Series:

$$e^{j\phi} = e^{j\phi}\Big|_{\phi=0} + \left(\frac{d}{d\phi}e^{j\phi}\right)\Big|_{\phi=0}\phi + \frac{1}{2}\frac{d^2}{d\phi^2}e^{j\phi}\Big|_{\phi=0} + \ldots$$

$$= 1 + j\phi - \frac{1}{2}\phi^2 - j\frac{1}{6}\phi^3 + \ldots$$

$$\cos\phi = \cos\phi\Big|_{\phi=0} + \frac{d}{d\phi}\cos\phi\Big|_{\phi=0}\phi + \frac{1}{2}\frac{d^2}{d\phi^2}\cos\phi\Big|_{\phi=0}\phi^2 + \ldots$$

$$= 1 + 0\cdot\phi - \frac{1}{2}\phi^2 + \ldots$$

$$j\sin\phi = j\sin\phi\Big|_{\phi=0} + j\frac{d}{d\phi}\sin\phi\Big|_{\phi=0}\phi + j\frac{1}{2}\frac{d^2}{d\phi^2}\sin\phi\Big|_{\phi=0}\phi^2 + \ldots$$

$$= j0 + j\phi + 0 - j\frac{1}{6}\phi^3 + \ldots$$

For $\frac{d}{dt}y(t) = \lambda y(t) + \gamma u(t)$

$u(t) = \underline{U} e^{j\omega t}$ ← easier to analyze

Assume $y(t) = \underline{Y} e^{j\omega t}$

$$\frac{d}{dt}\underline{Y}e^{j\omega t} = \lambda\underline{Y}e^{j\omega t} + \gamma\underline{U}e^{j\omega t}$$

$$j\omega\underline{Y}\cancel{e^{j\omega t}} = \lambda\underline{Y}\cancel{e^{j\omega t}} + \gamma\underline{U}\cancel{e^{j\omega t}}$$

$$(j\omega - \lambda)\underline{Y} = \gamma\underline{U}$$

$$\underline{Y} = \left(\frac{\gamma}{j\omega - \lambda}\right)\underline{U}$$

So

If $\frac{d}{dt} y(t) = \lambda y(t) + \gamma u(t)$

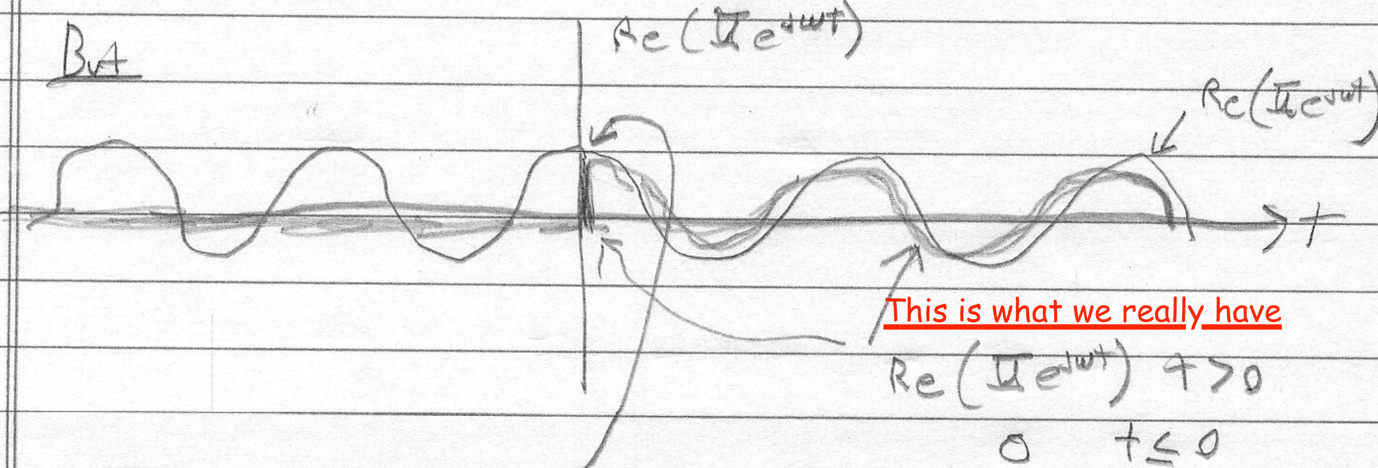
And $u(t) = \underline{U} e^{j\omega t}$ for all time $-\infty < t < \infty$

This is unrealistic

Then $y(t) = \underline{Y} e^{j\omega t}$

$$\underline{Y} = \underbrace{\frac{\gamma}{j\omega - \lambda}}_{\substack{\text{Frequency} \\ \text{Response} \\ H(j\omega)}} \underline{U}$$

But



Initial Conditions: right after input turns on

ZIR $\frac{d}{dt} y(t) = \lambda y(t)$

$y(t) = \alpha e^{\lambda t} \Rightarrow \frac{d}{dt} y(t) = \alpha \lambda e^{\lambda t} = \lambda y(t)$ ✓

check

Summing with response to input

$$y(t) = \underbrace{\alpha e^{\lambda t}} + H(j\omega) e^{j\omega t}$$

$\lim_{t \to \infty} \alpha e^{\lambda t} \to 0$ **if** $Re(\lambda) < 0$

* **Sinusoidal Steady State** $y(t) = H(j\omega) e^{j\omega t}$ t large enough

Example Velocity proportional Control

$$\underbrace{\frac{d}{dt}\underbrace{V(t)}_{\text{velocity}}}_{\text{acceleration}} = \underbrace{-\beta}_{\text{friction force}} V(t) + K_{p_1}\big(\underbrace{V_d(t)}_{\text{desired velocity}} - \underbrace{V(t)}_{\text{measured velocity}}\big)$$

Assuming: Mass = 1 (1·acceleration = force

Simplifying $\frac{d}{dt}V(t) = -\underbrace{(\beta + K_p)}_{\lambda} V(t) + \underbrace{K_p}_{\gamma} V_d(t)$

~~if t large out~~

Sinusoidal steady state $\longleftrightarrow$ λ $Re(\lambda) < 0$

If $V_d(t) = \underline{V}_d e^{j\omega t}$ **Assume** $V(t) = \underline{V} e^{j\omega t}$

$$j\omega \underline{V} = -(\beta + K_p)\underline{V} + K_p \underline{V}_d$$

$$\underline{V} = \frac{K_p}{j\omega + (\beta + K_p)} \underline{V}_d$$

For a good control system: $V(t) \simeq V_d(t)$

$\Rightarrow$ $\underline{V}$ should be $\simeq \underline{V}_d$

is it?

I) V close to V_d

In s.s.s. $\left(\begin{array}{c}\text{Exists because} \\ Re(-(B+K_p)) \\ < 0\end{array}\right)$

$$V = H(j\omega) V_d$$

$$H(j\omega) = \frac{K_p}{j\omega + (B+K_p)}$$

Two Plots

$$|H(j\omega)| \; , \quad \measuredangle H(j\omega)$$

$H(j\omega)$ is a complex number $= M(\omega) e^{j\phi(\omega)}$

$$M(\omega) = \left| \frac{K_p}{j\omega + (B+K_p)} \right| \qquad \phi(\omega) = \text{angle}\left(\frac{K_p}{j\omega + (B+K_p)}\right)$$

Aside ratios of complex numbers $z_i = a_i + jb_i$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \qquad |z_1 \cdot z_2| = |z_1| |z_2|$$

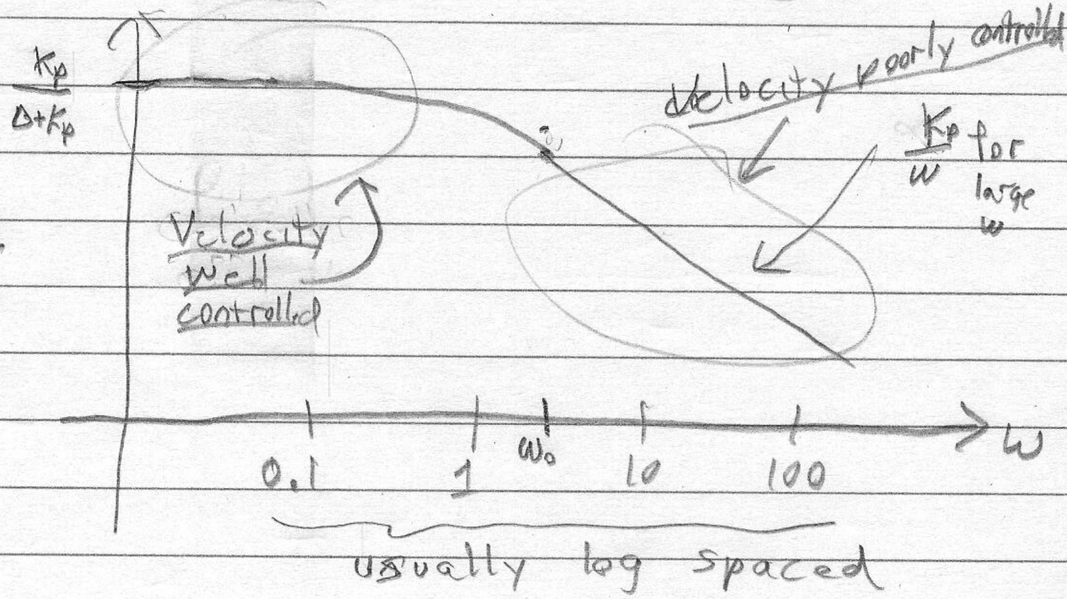
(Just use properties of polar form $z_i = M_i e^{j\phi_i}$)

$$\measuredangle \frac{z_1}{z_2} = \measuredangle z_1 - \measuredangle z_2$$

$$\measuredangle z_1 z_2 = \measuredangle z_1 + \measuredangle z_2$$

Magnitude Plot $\quad H(j\omega) = \dfrac{K_p}{j\omega + (B+K_p)}$

$|H(j\omega)|$



If $K_p >> B$

$|H(j\omega)| \cong 1$

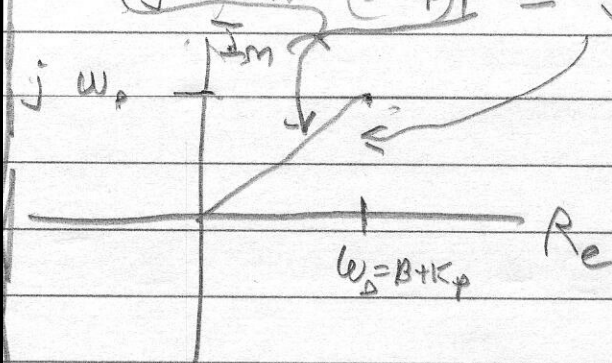
for $\omega < \omega_0$

Aside

$$\omega_0 \equiv B + K_p$$

$$\left| \frac{K_p}{j\omega_0 + B + K_p} \right| = \frac{K_p}{|j(B+K_p) + (B+K_p)|} = \frac{1}{\sqrt{2}} \frac{K_p}{B+K_p}$$

$$|j(B+K_p) + (B+K_p)| = \sqrt{2}\,|B+K_p|$$



Increasing K_p

$$\underbrace{\frac{K_p}{B+K_p} \to 1}_{\text{Better steady state}} \quad \& \quad \omega_0 \uparrow$$

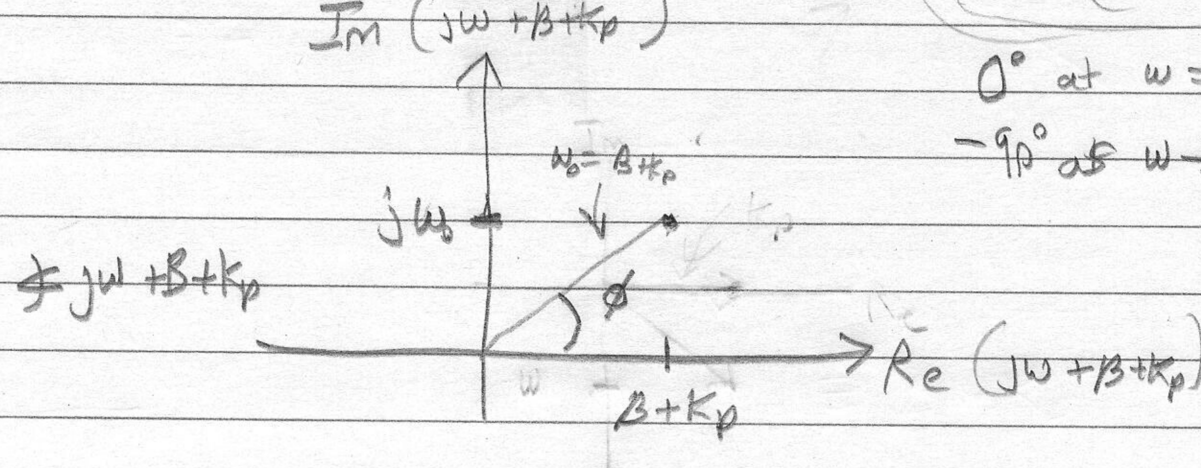
so well controlled for higher frequencies

Angle (Phase) Plot

$$\angle H(j\omega) = \angle \frac{K_p}{j\omega + (\beta + K_p)} = \underbrace{\angle K_p}_{=0^\circ} - \angle(j\omega + \beta + K_p)$$

0° at $\omega = 0$

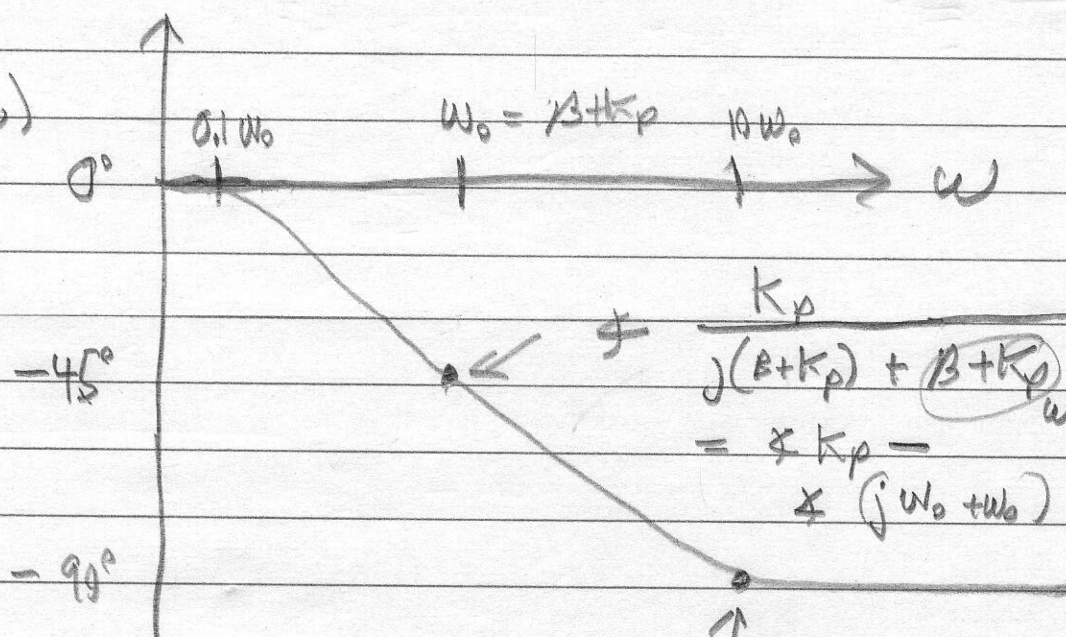
-90° as $\omega \to \infty$



0° $\angle H(j\omega)$

$= \angle K_p -$

$\angle(j\omega + K_p + \beta)$



$$\angle \frac{K_p}{j(\beta + K_p) + \underbrace{\beta + K_p}_{\omega_0}}$$

$= \angle K_p -$

$\angle (j\omega_0 + \omega_0)$

$$\simeq \frac{K_p}{j\omega}$$

$$= -j\frac{K_p}{\omega} = -90^\circ$$

Phase shift is Bad

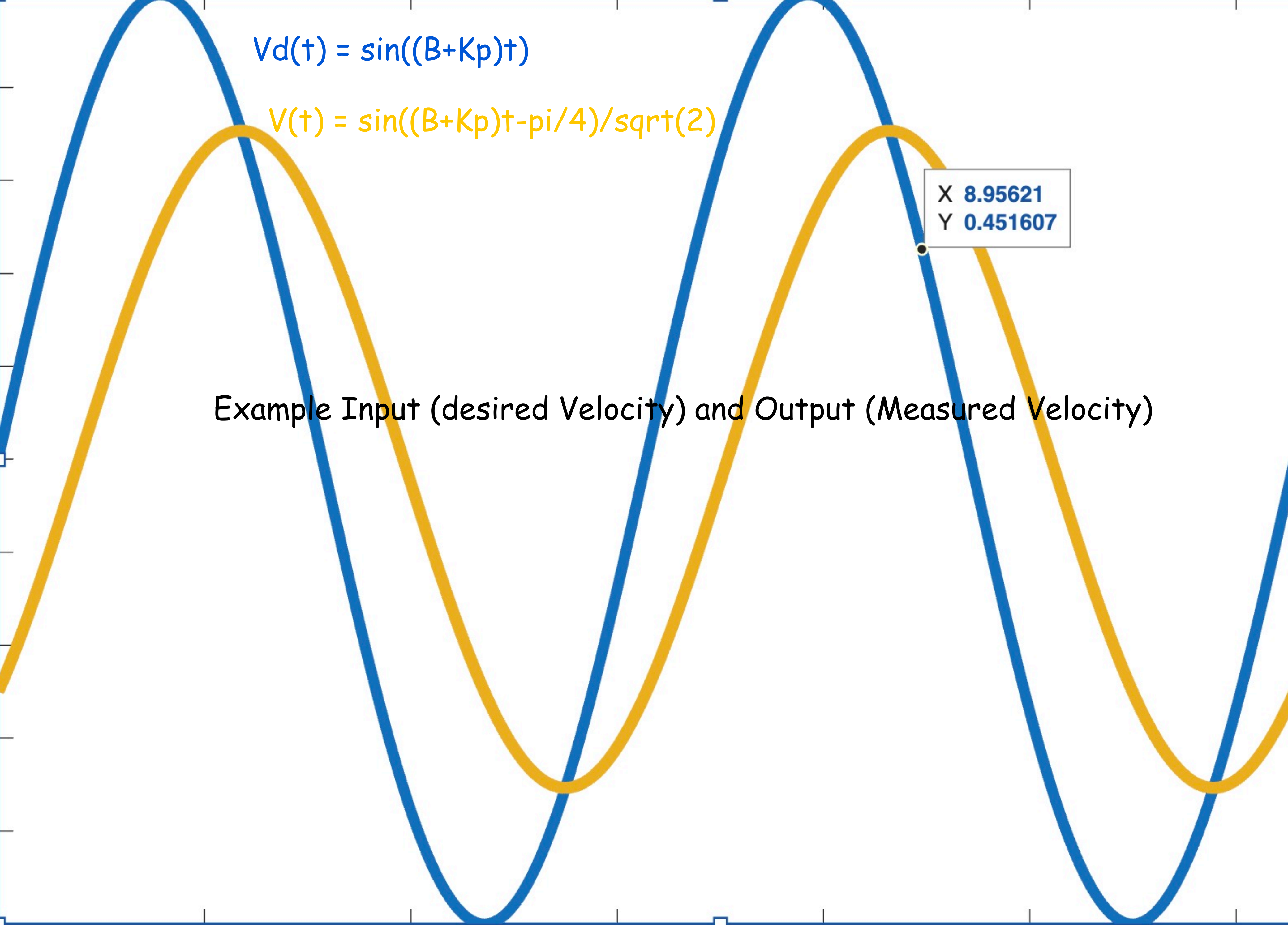
at $\omega = 10\,\omega_0$

output in shifted

90° from input.

Increasing K_p

ω_0 increases so



Vd(t) = sin((B+Kp)t)
V(t) = sin((B+Kp)t-pi/4)/sqrt(2)
X 8.95621
Y 0.451607
Example Input (desired Velocity) and Output (Measured Velocity)