6.3100/2 3/11/26

Last time

$$\frac{d}{dt}\big(f(t)\big) \Longleftrightarrow s\,F$$

operator   function of time   operator symbol   function of time

$$\int f(t) \Longleftrightarrow \frac{1}{s}\,F$$

integral

Key Identity

Given

$$X = \frac{\overbrace{b_{\tilde{L}} s^{\tilde{L}} + b_{\tilde{L}-1} s^{\tilde{L}-1} + \dots b_1 s + b_0}^{n(s)}}{\underbrace{a_L s^L + \dots a_1 s + a_0}_{d(s)}}\,U$$

numerator polynomial $n(s)$

$d(s)$ denominator polynomial

Then

$$a_L \frac{d^L}{dt^L} x(t) + a_{L-1} \frac{d^{L-1}}{dt^{L-1}} x(t) + \dots + a_1 \frac{d}{dt} x(t) + a_0 x(t)$$

$$= b_{\tilde{L}} \frac{d^{\tilde{L}}}{dt^{\tilde{L}}} u(t) + \dots + b_0 u(t)$$

ZIR

$$a_L \frac{d^L}{dt^L} x(t) + \dots + a_0 x(t) = 0$$

Assume $x(t) = Ae^{\lambda t} \Rightarrow \frac{d}{dt} x(t) = \lambda A e^{\lambda t}$

then $a_L \lambda^L \cancel{Ae^{\lambda t}} + a_{L-1} \lambda^{L-1} \cancel{Ae^{\lambda t}} + \dots + a_0 \cancel{Ae^{\lambda t}} = 0$

$$a_L \lambda^L + a_{L-1} \lambda^{L-1} + \dots + a_1 \lambda + a_0 = 0$$

Denote $\lambda_1, \ldots, \lambda_L$ as the L solns of

$$\sum_{l=0}^{L} a_l \lambda^l = 0$$

Then the ZIR is

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \ldots A_L e^{\lambda_L t}$$

Depend on $\underbrace{x(0),\ \frac{d}{dt}x(t)\Big|_{t=0}, \ldots \frac{d^{L-1}}{dt^{L-1}}x(t)\Big|_{t=0}}_{\text{Initial conditions}}$

Aside $A_1, \ldots A_L$ could be computed from almost any set of points $x(t_1), \ldots x(t_L)$

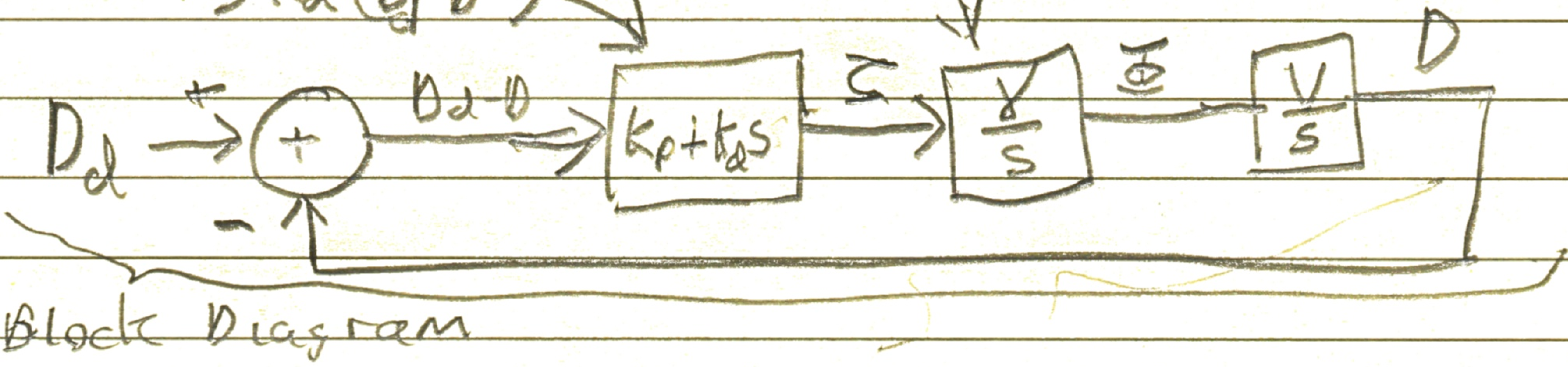
Ex Line (or wall) follower with P.D. control

desired distance — angle — distance

$$c(t) = K_p(d_d(t) - d(t)) + K_d \frac{d}{dt}(d_d(t) - d(t)) \qquad \frac{d}{dt}\theta(t) = \gamma c(t) \qquad \frac{d}{dt}d(t) = V\theta(t)$$

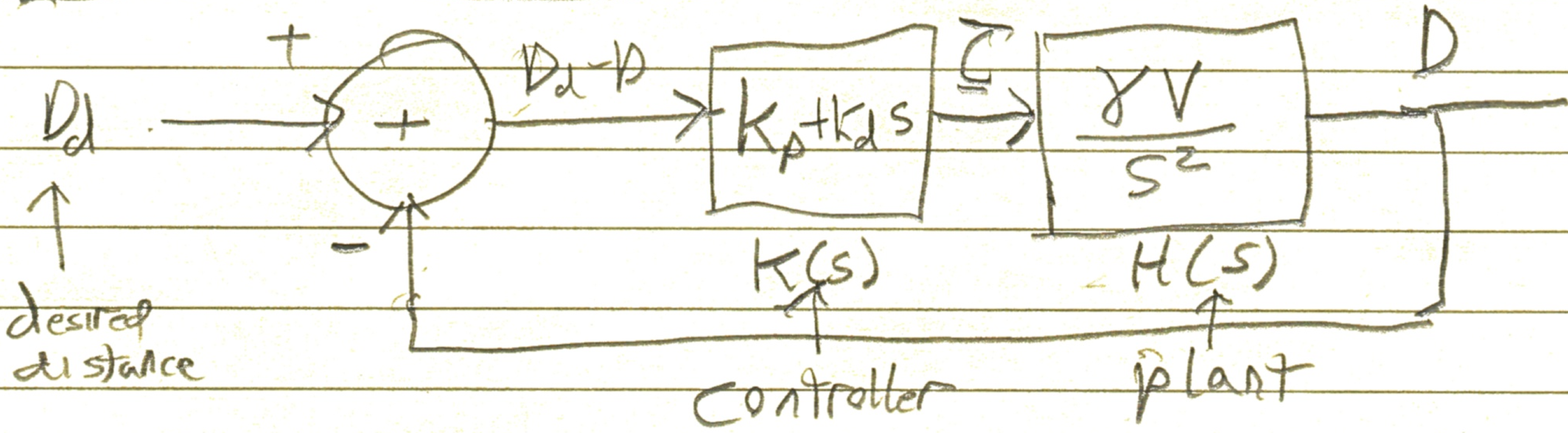
controller command — angle

$$C = K_p(D_d - D) + sK_d(D_d - D) \qquad \Phi = \frac{\gamma}{s} C \qquad D = \frac{V}{s}\Phi$$



Block Diagram

Line Follower Block Diagram



From Block Diagram

$$D = K(s)\,H(s)\,(D_d - D)$$

$$= (K_p + K_d s)\left(\frac{\gamma}{s}\frac{V}{s}\right)(D_d - D)$$

$$D = \underbrace{\frac{K(s)\,H(s)}{1 + K(s)\,H(s)}}_{\text{Blacks Formula}} D_d \longleftarrow G(s)$$

$$D = \frac{(K_p + K_d s)\left(\frac{\gamma V}{s^2}\right)}{1 + (K_p + K_d s)\left(\frac{\gamma V}{s^2}\right)} D_d$$

$$D = \underbrace{\frac{K_d \gamma V s + K_p \gamma V}{s^2 + K_d \gamma V s + K_p \gamma V}}_{G(s) \text{ for line follower}} D_d$$

Cross Multiplying

$$(s^2 + K_d \gamma V s + K_p \gamma V)\,D = (K_d \gamma V s + K_p \gamma V)\,D_d$$

de-operating

$$\frac{d^2}{dt^2} d(t) + K_d \gamma V \frac{d}{dt} d(t) + K_p \gamma V d(t) = K_d \gamma V \frac{d}{dt} d_d(t) + K_p \gamma V d_d(t)$$

ZIR for P.D. controlled line follower

$$\frac{d^2}{dt^2}d(t) + K_d \gamma V \frac{d}{dt}d(t) + K_p \gamma V d(t) = 0$$

Assuming $d(t) = Ae^{\lambda t}$ $\frac{d}{dt}(dt) = \lambda A e^{\lambda t}$

$$\lambda^2 A\cancel{e^{\lambda t}} + K_d \gamma V \lambda A \cancel{e^{\lambda t}} + K_p \gamma V A \cancel{e^{\lambda t}} \underset{\text{ZIR!}}{=} 0$$

Only true if

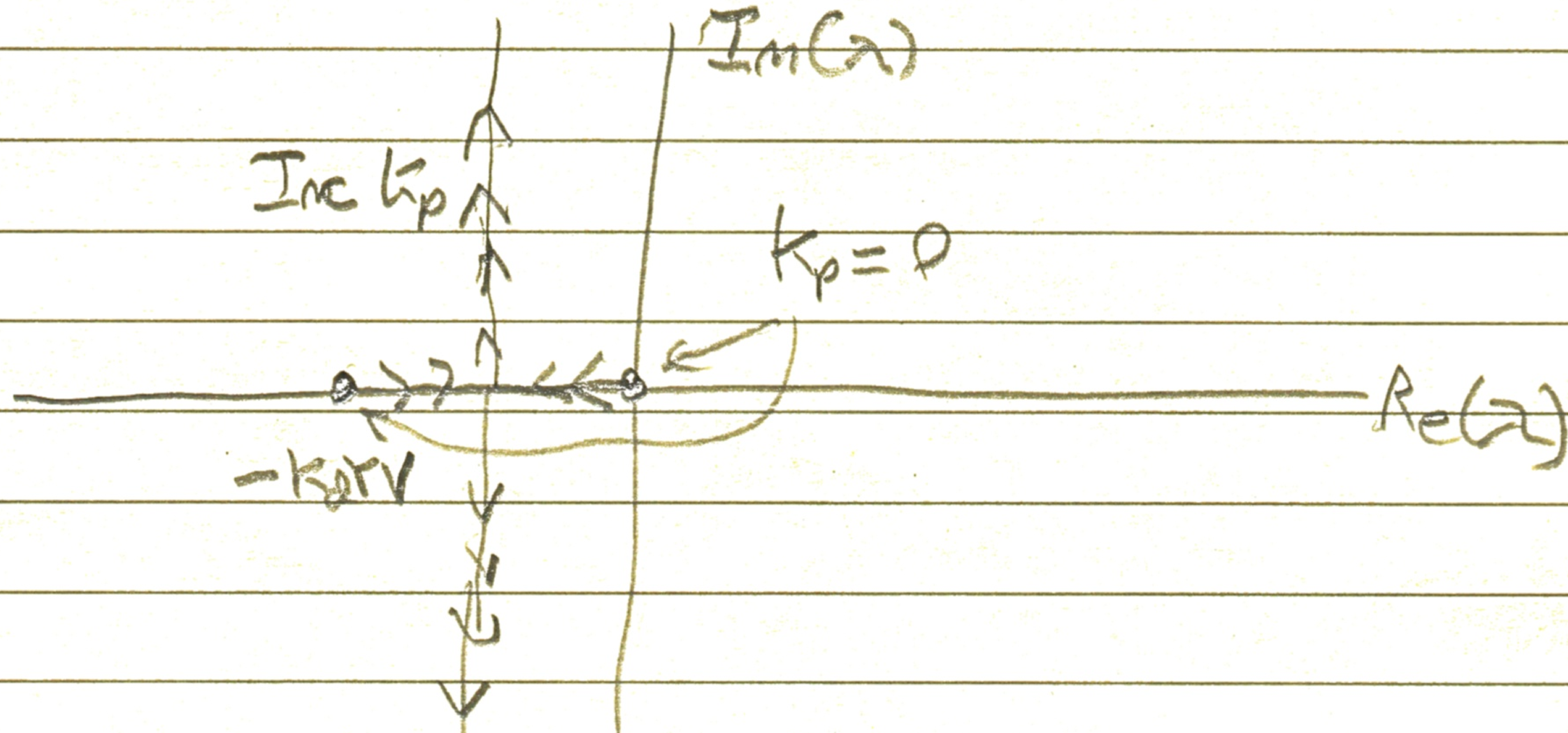
Compare to $d(s)$ (denominator poly) $\Longrightarrow$ $\lambda^2 + K_d \gamma V \lambda + K_p \gamma V = 0$

$$\lambda_1, \lambda_2 = -\frac{K_d \gamma V}{2} \pm \sqrt{\frac{(K_d \gamma V)^2}{4} - K_p \gamma V}$$

$$d_{ZIR}(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

Stability $\longrightarrow \lim_{t \to \infty} d_{ZIR}(t) = 0$ if $Re(\lambda_1) < 0$ & $Re(\lambda_2) < 0$

λ_1, λ_2 Versus K_p given $K_d > 0$



$Re(\lambda_1) < 0$ & $Re(\lambda_2) < 0$ if $K_d > 0$

For Any K_p!

In General

Control

$$K(s) = \frac{n_K(s)}{d_K(s)} \begin{array}{l} \leftarrow \text{numerator poly in } s \\ \leftarrow \text{denominator poly in } s \end{array}$$

Example PID

$$K(s) = K_p + K_d s + \underbrace{K_i \frac{1}{s}}_{\text{Integral}} = \frac{\overbrace{K_p s + K_d s^2 + K_i}^{n_K(s)}}{\underbrace{s}_{d_K(s)}}$$

Plant

$$H(s) = \frac{n_H(s)}{d_H(s)}$$

Example

$$H(s) = \frac{\gamma}{s}\frac{V}{s} = \frac{\gamma V \leftarrow n_H(s)}{s^2 \leftarrow d_H(s)}$$

Blacks formula

$$G(s) = \frac{K(s)H(s)}{1 + K(s)H(s)} = \frac{\frac{n_K(s)}{d_K(s)}\frac{n_H(s)}{d_H(s)}}{1 + \frac{n_K(s)}{d_K(s)}\frac{n_H(s)}{d_H(s)}}$$

$$= \frac{\overbrace{n_K(s)\, n_H(s)}^{n_g(s)}}{\underbrace{d_K(s)\, d_H(s) + n_K(s)\, n_H(s)}_{d_g(s)}}$$

Given

$$G(s) = \frac{n_g(s)}{d_g(s)} = \frac{b_{\tilde{L}} s^{\tilde{L}} + \dots b_1 s + b_0}{a_L s^L + \dots a_1 s + a_0}$$

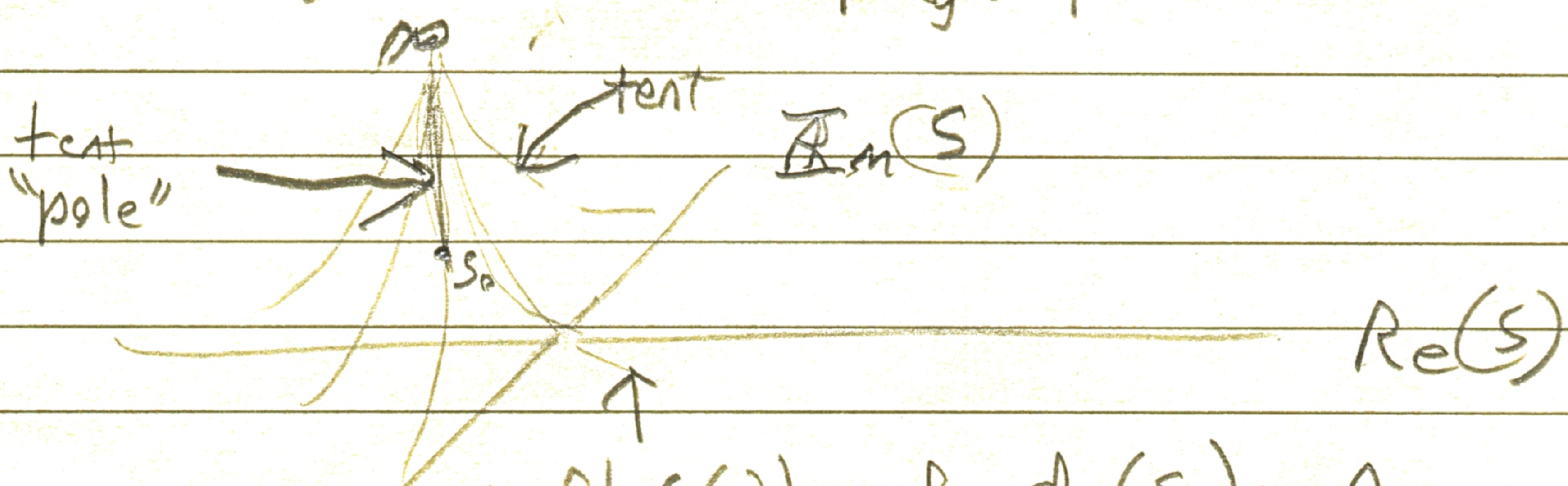
Natural freqs λ s.t. if $s = \lambda$

$$d_g(s)\Big|_{s=\lambda} = 0$$

roots of $d_g(s)$ = nat freqs

Also Called Poles Why?

$$|G(s)| = \frac{|n_g(s)|}{|d_g(s)|}$$



plot of $|G(s)|$ if $d_g(s_0) = 0$
versus s

If Re(poles of $G(s)$) < 0

System is stable!

So

What about roots of $n_g(s)$?

Assume $d(t) = 0$ what $d_d(t) = A e^{\lambda t}$ leaves $d(t)$ unchanged

Robot Example

$$0 = K_d \gamma V \frac{d}{dt} A e^{\lambda t} + K_p \gamma V A e^{\lambda t}$$

$$0 = K_d \gamma V \lambda \cancel{A e^{\lambda t}} + K_p \gamma V \cancel{A e^{\lambda t}}$$

$$\lambda = -\frac{K_p \gamma V}{K_d \gamma V}$$

So roots of $n_g(s)$ are called zeros!

$$G(s) = \frac{n_g(s)}{d_g(s)}$$

roots = zeros

roots = poles = nat freqs ← tests stability

What about frequency response

If $d_d(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$

What does $G(s)$ tell us

$$G(s) = \frac{K(s)H(s)}{1+K(s)H(s)} = \frac{n_g(s)}{d_g(s)}$$

Alternative to operator 's' derivation

Assume All variables are scaled $e^{j\omega t}$ ← Note $\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}$

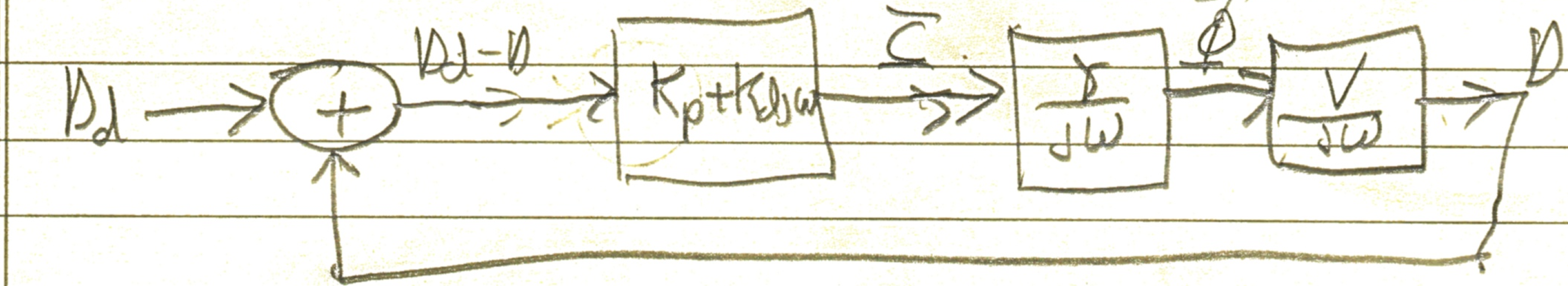
Ex $\quad \frac{d}{dt}\theta(t) = \gamma c(t)$

$$\frac{d}{dt}\underline{\Phi} e^{j\omega t} = j\omega \underline{\Phi} e^{j\omega t} = \gamma \underline{C} e^{j\omega t}$$

$$\underline{\Phi} = \frac{\gamma}{j\omega}\underline{C}$$

$$\frac{d}{dt} d(t) = V\theta(t) \Rightarrow$$

$$j\omega D \cancel{e^{j\omega t}} = V\underline{\Phi}\cancel{e^{j\omega t}} \Rightarrow D = \frac{V}{j\omega}\underline{\Phi}$$



$$D = G(s)\Big|_{s=j\omega} D_d = G(j\omega) D_d$$

OR

$$D = \frac{K(j\omega)\,H(j\omega)}{1 + K(j\omega)\,H(j\omega)}\,D_d$$

$G(j\omega) \equiv$ freq Response

Learn from freq Response

If $|K(j\omega)\,H(j\omega)| >> 1$

$D \simeq D_d$ at frequency ω

Good tracking

How do we make $K(j\omega)\,H(j\omega)$ Big?

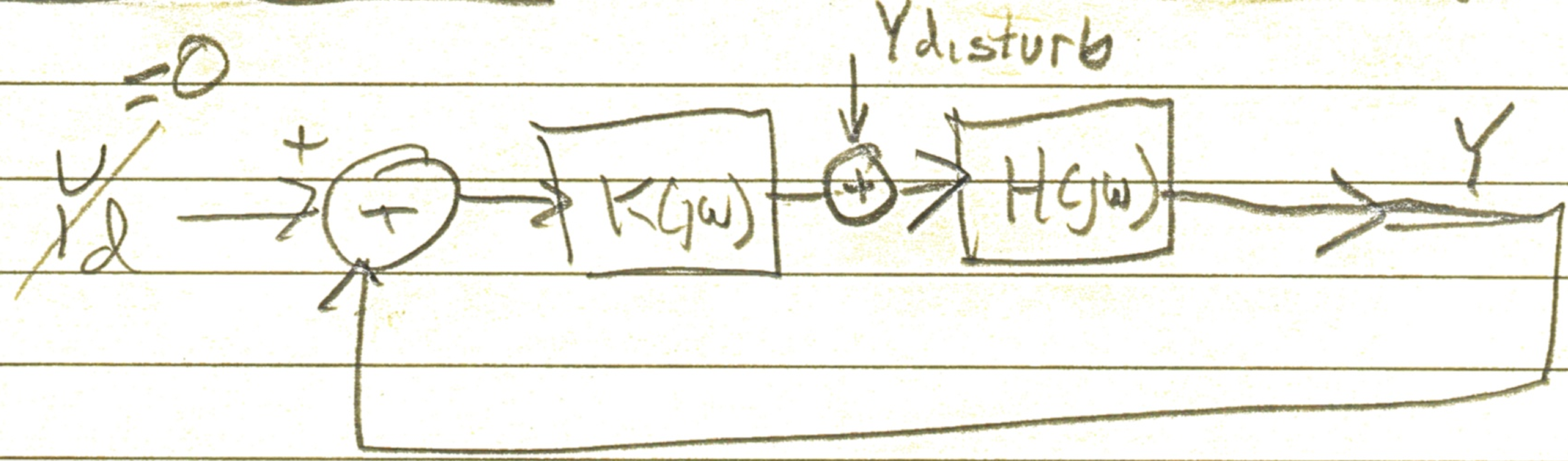
$$K(j\omega) = K_p + K_d\, j\omega + K_i \frac{1}{j\omega}$$

$\lim_{\omega \to \infty} K(j\omega) \to \infty$ if $K_d \neq 0$

$\lim_{\omega \to 0} K(j\omega) \to \infty$ if $K_i \neq 0$

Disturbances

$Y(j\omega) = G_d(j\omega)\, Y_d(j\omega)$



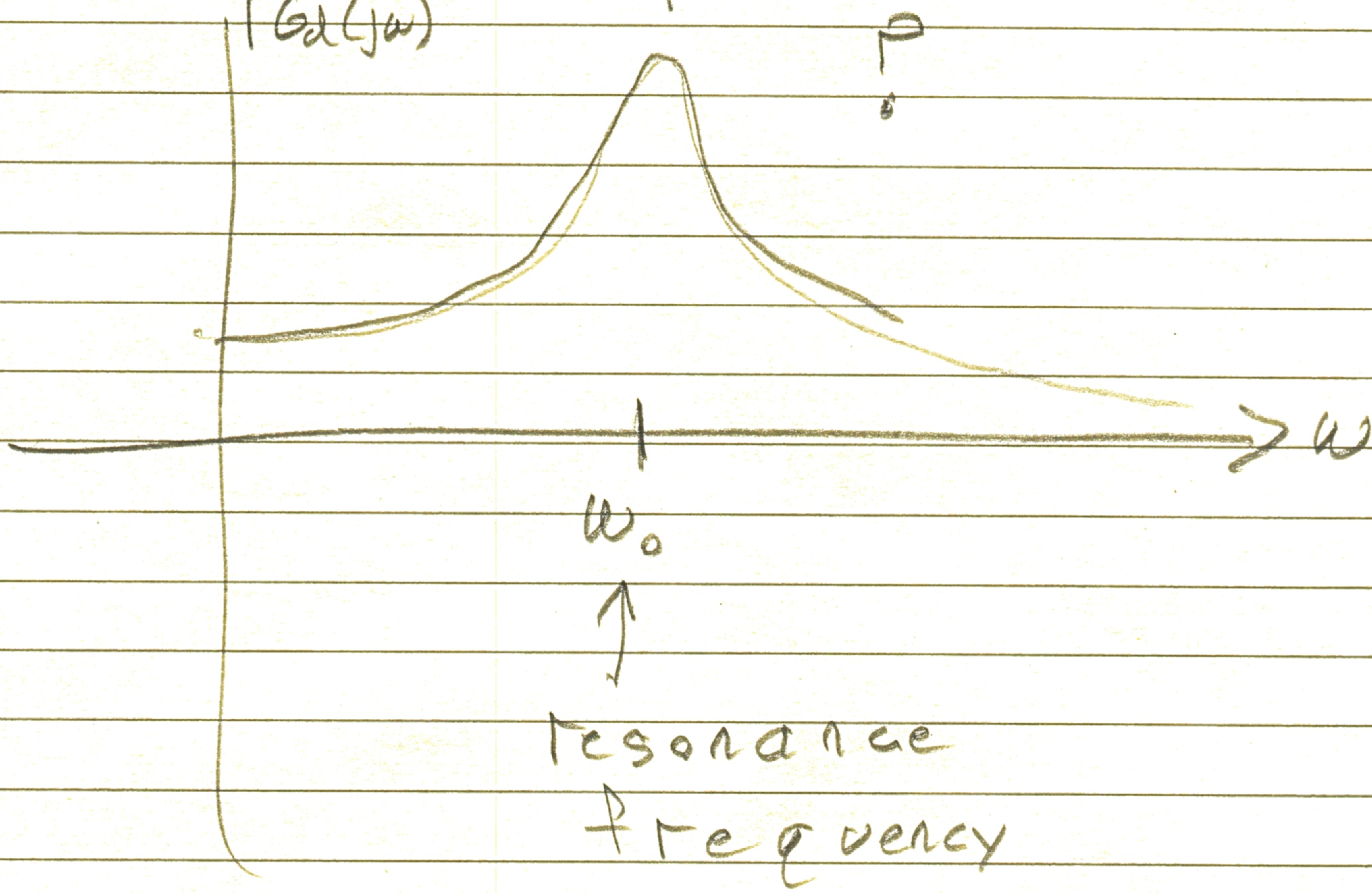
$G_d(j\omega) = ?$

Want $G_d(j\omega) = 0$
to reject all disturbances

For Prop Arm in Lab 3A

(Any or nearly unstable controllers)

What was $|G_d(j\omega)|$?



Can we design controllers that minimize $|G_d(j\omega)|$?