Dynamic System Modeling and Control Design

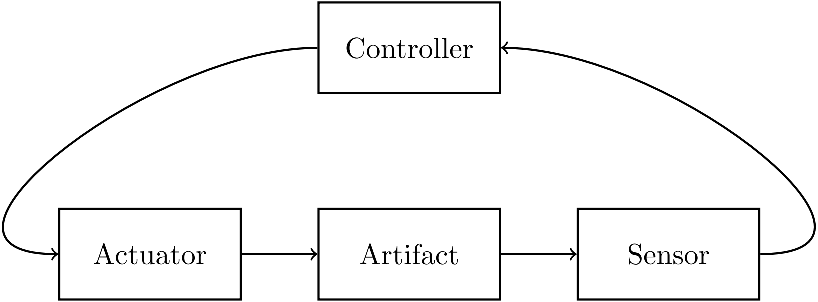
Linearity, Time Invariance, Parameter Identification

Feb 9th, 2026

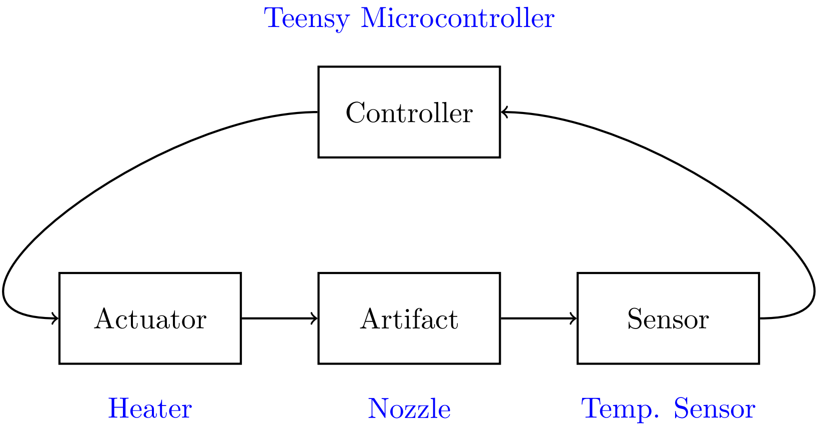
Outline

1. Recap of Last Lecture
2. Linearity and Time Invariance
3. Estimating System Parameters λ & γ

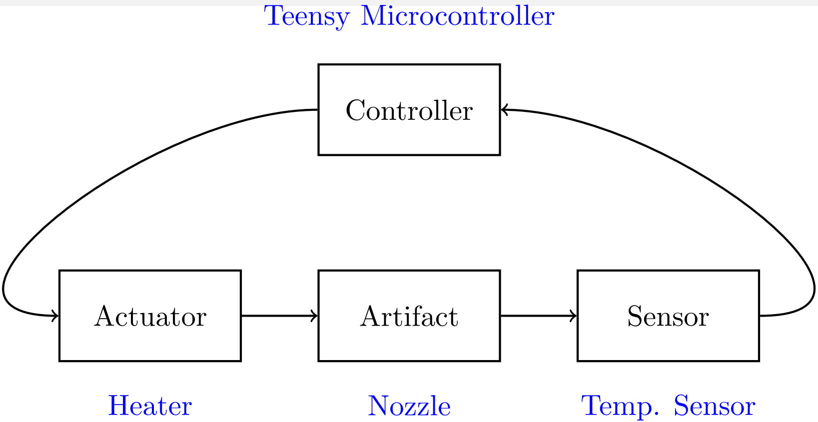
Recap: Generic Control System



Recap: Generic Control System



Recap: Generic Control System



$$T_m[n] = T_m[n-1] + \Delta T \gamma_{th} u[n-1] \underbrace{-\Delta T \beta T_m[n-1]}_{\text{heat loss term}}$$

Recap: General Form of First Order Difference Equation (FODE)

The general form of a first order difference equation:

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

Notes on the general form:

- Our goal is to solve for $y[n]$,
- $u[n]$ is the input or driving function we set,
- λ, γ are system parameters.

Recap: General Solutions of FODE

From the general form,

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

we found the general solution, for arbitrary n, to be:

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$$

Additionally, two special cases:

- Zero Input Response: If $u[n] = 0 \; \forall n: \; y[n] = \lambda^n y[0]$
- Zero State Response: If $y[0] = 0: \; y[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$

Recap: Steady State Response

If $u[n] = u_0 \ \forall n$ (i.e., constant input) AND $|\lambda| < 1$, then...

$$\lim_{n\to\infty} y[n] := y[\infty] = \frac{\gamma}{1-\lambda} u_0.$$

For example, for the heating example with proportional control, with $u[n] = K_p(T_d[n] - T_m[n])$ and $|\lambda| = |1 - \Delta T \gamma_{th} K_p| < 1$,

$$T_m[\infty] = \frac{\gamma_{th} K_p}{\gamma_{th} K_p + \beta} T_{d_0}$$

Post Lecture Notes

6.310/2 2/09/26 (1)

Today

- Properties of 1^{st}-order Lin Dff Eqns
- Tests for estimating params

Gen 1^{st}-Order DE $y[n] = \lambda y[n-1] + \gamma u[n-1]$

Gen Sol: $y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$

Zero input response → **ZIR** $(u[n]=0)$ $y_{ZIR}[n] = \lambda^n y[0]$

Zero state response ZSR $(y[0]=0)$ $y_{ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$

Property I $y[n] = y_{ZIR}[n] + y_{ZSR}[n]$

Property II Linearity and Time Invariance of ZSR

If

$$y_{A_{ZSR}}[n] = \gamma \sum_{n=0}^{n-1} \lambda^{(n-m-1)} u_A[m]$$

$$y_{B_{ZSR}}[n] = \gamma \sum_{n=0}^{m-1} \lambda^{(n-m-1)} u_B[m]$$

→ **Then Given** $u[n] = \alpha u_A[n-N_A] + \beta u_B[n-N_B]$

$$y_{ZSR}[n] = \alpha y_{A_{ZSR}}[n-N_A] + \beta y_{B_{ZSR}}[n-N_B]$$

Assuming $N_A \geq 0$, $N_B \geq 0$, $u_{A,B}[n] = 0$ $\underline{n<0}$

Proof

$$y_{ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{n-m-1} \left(\alpha u_A[n-N_A] + \beta u_B[n-N_B]\right)$$

$$= \alpha \gamma \sum_{m=0}^{n-1} \lambda^{n-m-1} u_A[m-N_A] + \beta\gamma \sum_{m=0}^{n-1} \lambda^{n-m-1} u_B[m-N_B]$$

$$\underbrace{\quad}_{= y_{A\,ZSR}[n-N_A]\,?} \qquad \underbrace{\quad}_{y_{B\,ZSR}[n-N_B]}$$

Proof Consider $\gamma \sum_{m=0}^{n-1} \lambda^{n-m-1} u_A[m-N_A]$

$= 0$ for $n \le N_A$ $\left(u_A[m-N_A] = 0 \quad m < N_A\right)$

$$= \gamma \sum_{m=N_A}^{m=n-1} \lambda^{n-m-1} u_A[m-N_A] \qquad n \ge N_A$$

$$= \gamma \sum_{\tilde{m}=0}^{\tilde{m}=(n-N_A)-1} \lambda^{(n-N_A)-\tilde{m}-1} u_A[\tilde{m}]$$

$$= y_{A\,ZSR}[n-N_A]$$

Property III if $u[n] = 0 \quad \forall\ n \ge N$

$$y[n] = \lambda^{n-N} y[N]$$

Proof

$$y[N+1] = \lambda y[N] + \gamma \underbrace{u[N]}_{=0}$$

$$y[N+2] = \lambda y[N+1] + \gamma \underbrace{u[N+1]}_{=0}$$

$$= \lambda^2 y[N]$$

$\vdots$

Property IV Steady-State

If $|\lambda| < 1$ and $u[n] = u_0 \ \forall n \geq N$ (finite N)

$$\lim_{n\to\infty} y[n] \equiv y[\infty] = \frac{\gamma}{1-\lambda} u_0$$

Proof

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{n-m-1} u[m]$$

for $n \geq N$

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=N}^{n-1} \lambda^{n-m-1} u[m] + \gamma \sum_{m=0}^{N} \lambda^{n-m-1} u[m]$$

$$\lim_{n\to\infty} y[n] = \underbrace{\lim_{n\to\infty} \lambda^n y[0]}_{=0 \text{ if } |\lambda|<1} + \lim_{n\to\infty} \gamma \sum_{m=N}^{n-1} \lambda^{n-m-1} u_0 + \underbrace{\lim_{n\to\infty} \gamma \lambda^{n-N} \sum_{m=0}^{N} \lambda^{N-m-1} u[m]}_{=0 \text{ if } |\lambda|<1}$$

Note

$$\lim_{n\to\infty} y[n] = \lim_{n\to\infty} \gamma u_0 \sum_{m=N}^{n-1} \lambda^{n-m-1}$$

$$= \gamma u_0 \lim_{n\to\infty} \left(\sum_{k=0}^{n-N-1} \lambda^k \right)$$

$$= \sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}$$

$$= \frac{\gamma u_0}{1-\lambda}$$

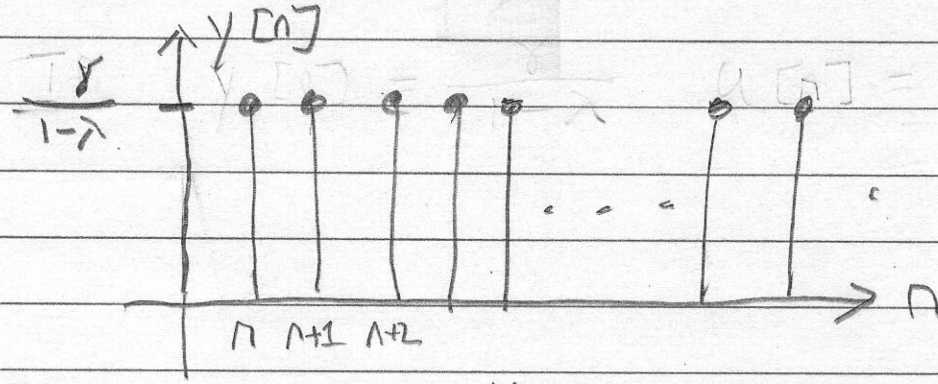
Using the Properties

Assuming $|\lambda| < 1$

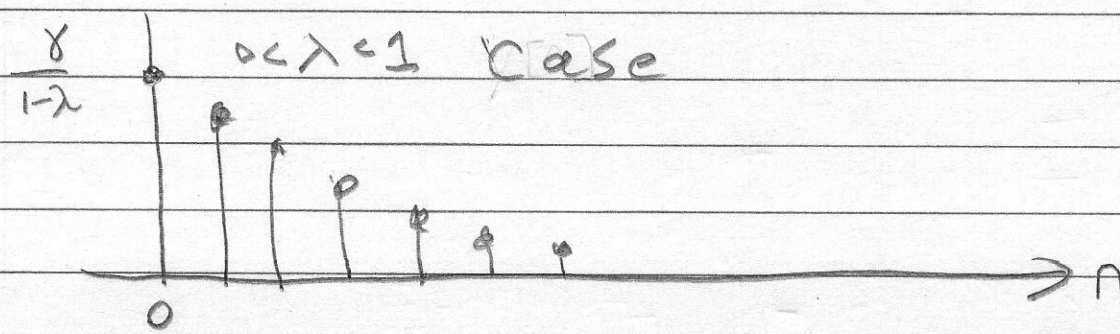
If $u[n] = 1 \ \forall n$ & $y[0] = \frac{\gamma}{1-\lambda}$

then $y[n] = \gamma/_{1-\lambda} \ \forall n$

(Initial cond matches S.S.)

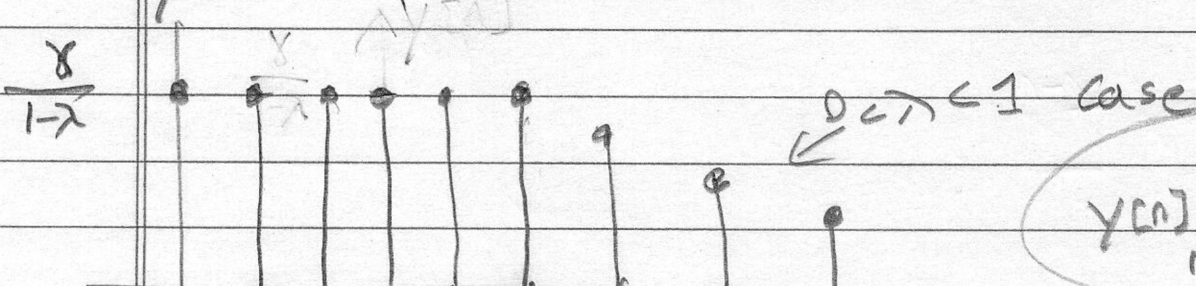


If $u[n] = 0 \ \forall n$ $y[0] = \frac{\gamma}{1-\gamma}$



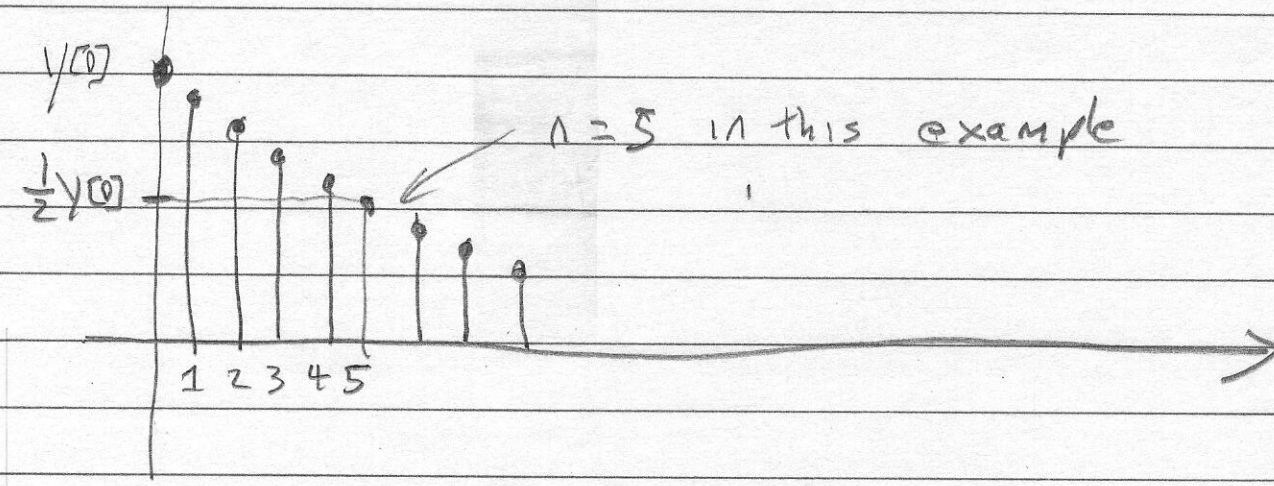
If $u[n] = 1$ $n < N$ $u[0] = 0$ $n \geq N$

and N is large enough to reach steady stat



$$y[n] = \lambda^{n-N}\left(\frac{\gamma}{1-\lambda}\right) \quad n > N$$

Estimating λ from ZIR $y[n] = \lambda^n y[0]$



find n_0 for which $y[n_0] \simeq \frac{1}{2} y[0]$

$$\frac{1}{2} y[0] = \lambda^{n_0} y[0]$$

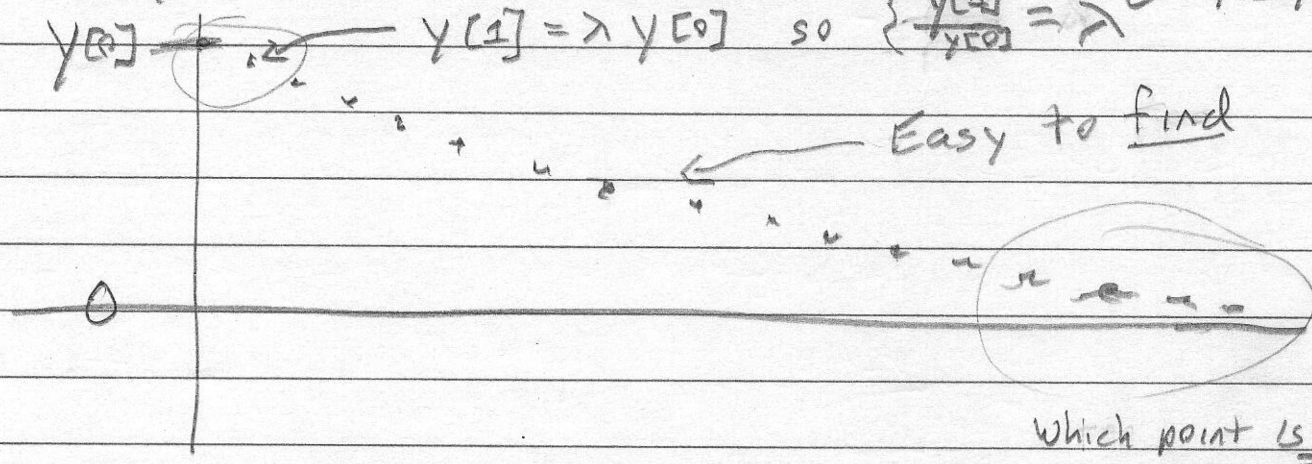
$$\lambda = \left(\frac{1}{2}\right)^{1/n_0}$$

Why use halfway?

$y[1] = \lambda y[0]$ so $\frac{y[1]}{y[0]} = \lambda$

Very sensitive to Noise in $y[1]$

typically 0.95 $y[1] \simeq y[0]$



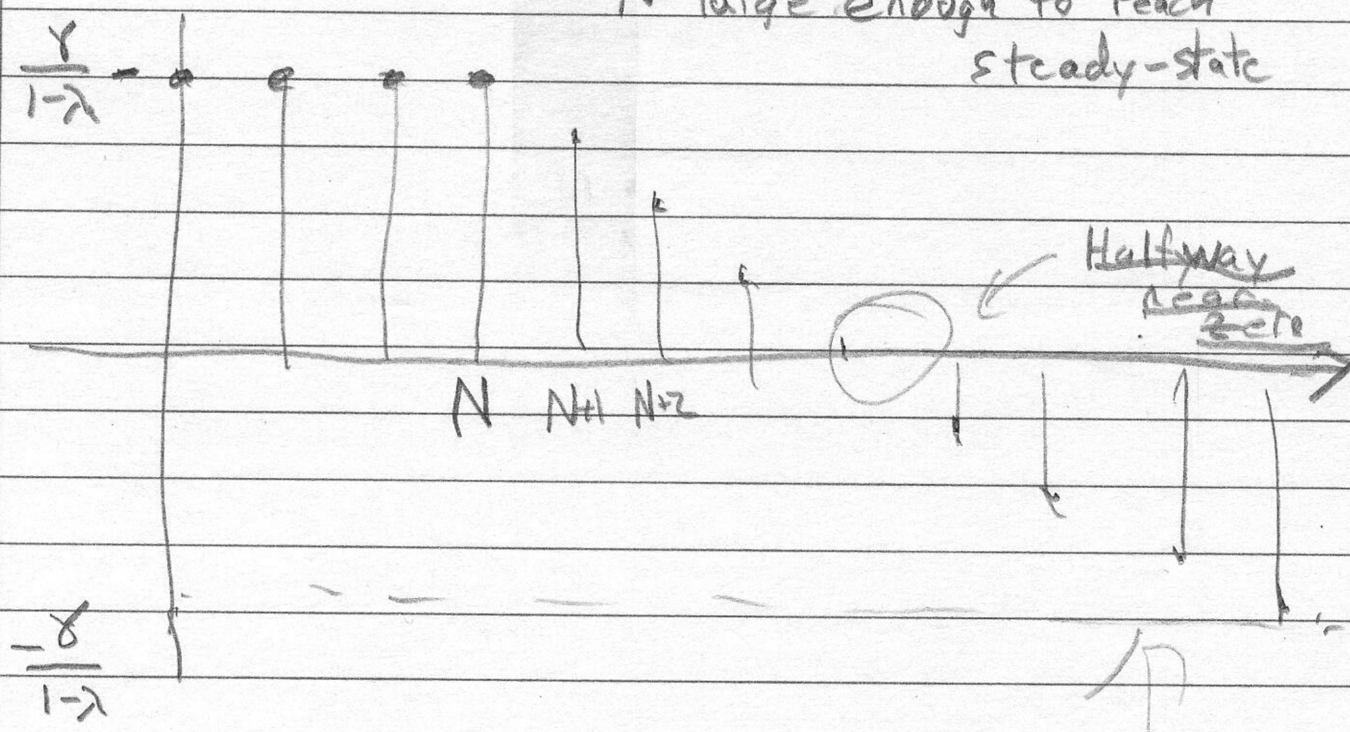
Which point is $0.1\, y[0]$?

"low slope"

Suppose

$u[n] = 1 \quad \forall\, n < N \qquad u[n] = -1 \quad \forall\, n \geq N$

N large enough to reach steady-state



By Linearity

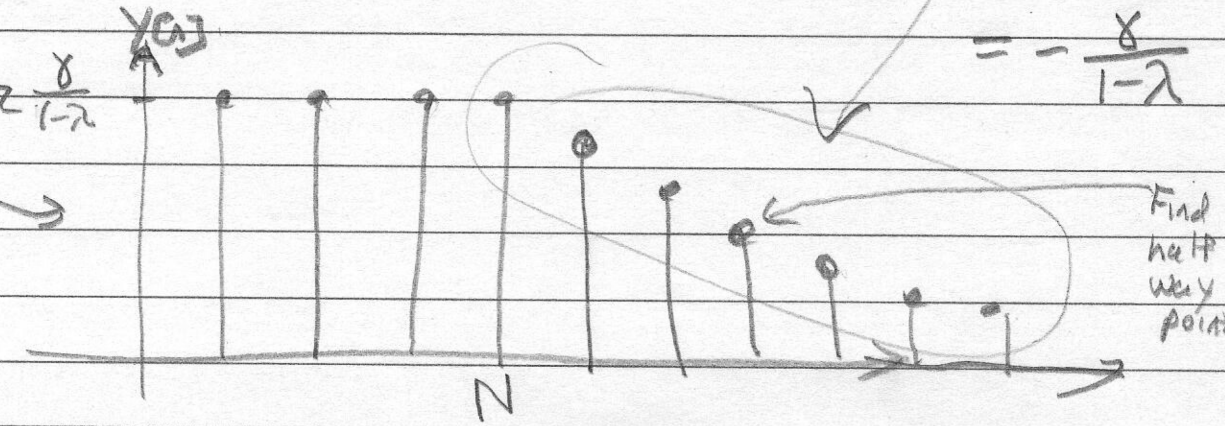
$u[n] = u_A[n] + u_B[n]$

$y[n] = y_A[n] + y_B[n]$

Same dynamic behavior

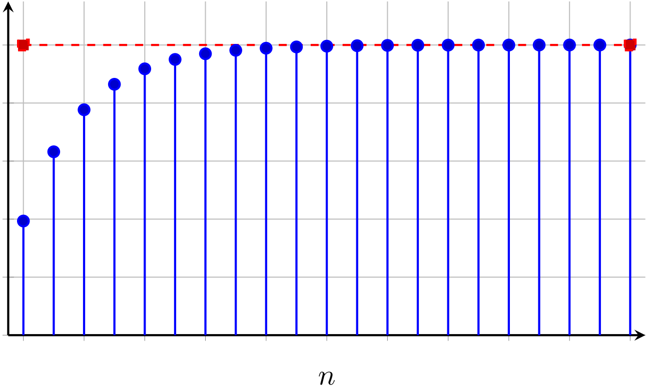
$$\begin{cases} u_A[n] = 2 & \forall\, n < N \\ u_A[n] = 0 & \forall\, n \geq N \end{cases}$$

$$u_B[n] = -1 = u_0 \quad \forall\, n \Rightarrow y_B[n] = \frac{\gamma}{1-\lambda} u_0 = -\frac{\gamma}{1-\lambda}$$



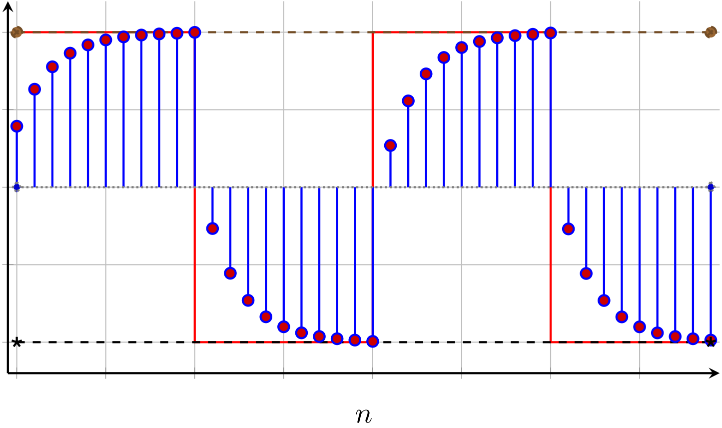
Unused Slides from Lecture

How do we go from this...



Today's First Objective

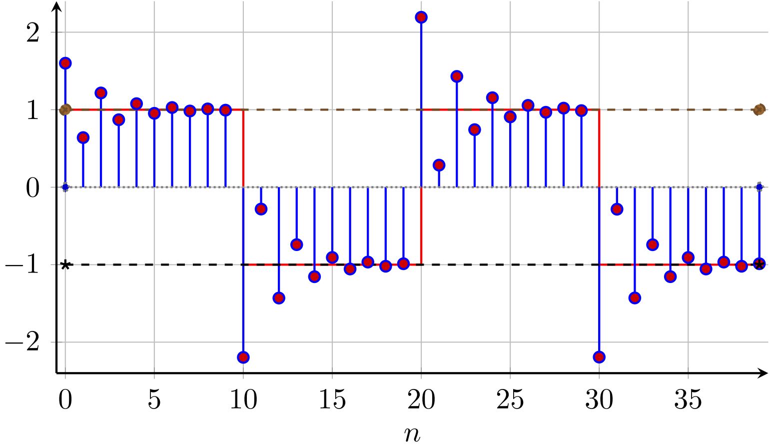
... to this??



Answer: Linearity and Time Invariance.

Today's First Objective

... or this??



Answer: Linearity and Time Invariance.

Today's Second Objective

We have an FOLDE:

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

What are λ and γ? How do we find them?

We will introduce how to estimate these system parameters today!

Property I: Decomposition of General Solution

Recall our two special cases:

- Zero Input Response: If $u[n] = 0 \; \forall n: \; y[n] = \lambda^n y[0]$
- Zero State Response: If $y[0] = 0: \; y[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$

Then we can decompose $y[n]$ into its ZIR and ZSR:

$$y[n] = \lambda^n y[0] + \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u[m]$$

$$\text{(I): } y[n] = y_{ZIR}[n] + y_{ZSR}[n]$$

Property II: Linearity of ZSR

Given two different input functions $u_A[n], u_B[n]$. Then,

$$y_{A,ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_A[m],$$
$$y_{B,ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_B[m].$$

If $u[n] = \alpha u_A[n] + \beta u_B[n]$, then

$$\text{(II): } y_{ZSR}[n] = \alpha y_{A,ZSR}[n] + \beta y_{B,ZSR}[n].$$

Property III: Time Invariance of ZSR

Given an input functions $u_A[n]$,

$$y_{A,ZSR}[n] = \gamma \sum_{m=0}^{n-1} \lambda^{(n-m)-1} u_A[m]$$

Assume $u_A[n] = 0 \; \forall n < 0$ and $N_A > 0$.

If $u[n] = u_A[n - N_A]$ then

$$\text{(II): } y_{ZSR}[n] = y_{A,ZSR}[n - N_A]$$

Property IV: An Aspect of Time Invariance of ZIR

If $u[n] = 0 \; \forall n \geq N$, then

$$\text{(III): } y[n] = \lambda^{n-N} y[N], \; \forall n > N.$$

Proof sketch:

$$\begin{aligned} y[N+1] &= \lambda y[N] + \gamma u[N] \\ y[N+2] &= \lambda y[N+1] + \gamma u[N+1] \\ &= \lambda^2 y[N] \\ y[N+3] &= \lambda^3 y[N] \\ &\vdots \end{aligned}$$

Steady State for General System (Using (I) - (VI))

Suppose that I have $|\lambda| < 1$ and an input function $u_1[n]$ defined by,

$$u_1[n] = 0,\ n < N$$
$$u_1[n] = 1,\ n \geq N.$$

with an initial state of $y_1[0] = 0$. What is $y_1[n]$?

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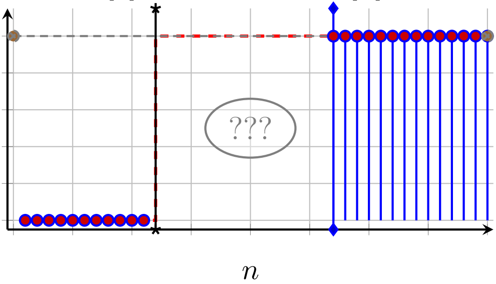

Let's find out using Properties (I)-(III)!

Steady State (Cont.)

Consider a system with

$$y_2[0] = \frac{\gamma}{1-\lambda},$$
$$u_2[n] = 1 \; n < N,$$
$$u_2[n] = 0 \; n \geq N.$$

What is $y_2[n]$ for $n > N$?

Steady State (Cont.)

Consider a system with

$$y_2[0] = \frac{\gamma}{1-\lambda},$$
$$u_2[n] = 1 \; n < N,$$
$$u_2[n] = 0 \; n \geq N.$$

What is $y_2[n]$ for $n > N$?
We can use Property (III): Time Invariance of ZIR:

$$y_2[n] = \frac{\gamma}{1-\lambda} \; n \leq N$$
$$y_2[N+1] = \lambda y_2[N] = \lambda \frac{\gamma}{1-\lambda}$$
$$y_2[N+n] = \lambda^{(n-N)} \frac{\gamma}{1-\lambda}$$

Steady State (Cont.)

Consider a system with

$$y_3[0] = \frac{\gamma}{1-\lambda},$$
$$u_3[n] = 1 \ \forall n,$$

What is $y_3[n]$?

Steady State (Cont.)

Consider a system with

$$y_3[0] = \frac{\gamma}{1-\lambda},$$
$$u_3[n] = 1 \ \forall n,$$

What is $y_3[n]$?
Since we initialized at steady state, and the input function $u_3[n]$ does not change, we will remain in steady state.

$$y_3[n] = \frac{\gamma}{1-\lambda}.$$

Steady State for General System (Using (I) - (III))

Recall that $|\lambda| < 1, u[n] = u_0 \ \forall n > N$, then $y[\infty] = \frac{\gamma}{1-\lambda} u_0$.
Suppose that I have $|\lambda| < 1$ and an input function $u_1[n]$ defined by,

$$u_1[n] = 0,\ n < N$$
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$$u_1[n] = 0,\ n < N$$
$$u_1[n] = 1,\ n \geq N.$$

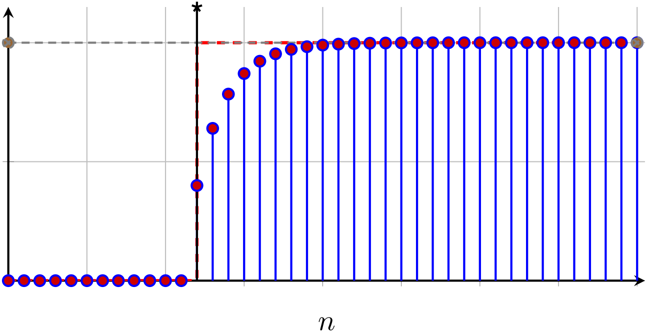
with an initial state of $y_1[0] = 0$. What is $y_1[n]$?
We can use Property (II): Linearity of ZSR

Since $u_1[n] = u_3[n] - u_2[n]$, we know that $y_1[0] = y_3[0] - y_2[0]$.
Therefore,

$$\begin{aligned} y_1[n] &= y_3[n] - y_2[n] \\ &= \frac{\gamma}{1-\lambda} - \lambda^{n-N} \frac{\gamma}{1-\lambda},\ n > N \\ &= \frac{\gamma}{1-\lambda}(1 - \lambda^{n-N}),\ n > N. \end{aligned}$$

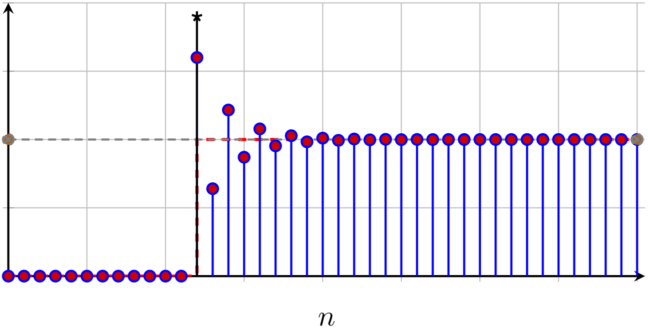
Now, we can fill in the gap...

$$y_1[n] = \frac{\gamma}{1-\lambda}(1-\lambda^{n-N}),\ n > N, \underline{0 < \lambda < 1}.$$



Now, we can fill in the gap...

$$y_1[n] = \frac{\gamma}{1-\lambda}(1-\lambda^{n-N}),\ n > N, \underline{-1 < \lambda < 0}.$$



Recall Today's Second Objective

We have this nice FODE:

$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

We can (experimentally) estimate λ, γ in many different ways.

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$$y[n] = \lambda y[n-1] + \gamma u[n-1] \qquad (\#1)$$

We can (experimentally) estimate λ, γ in many different ways.

In particular, we have two unknowns. Let's find two equations and solve.

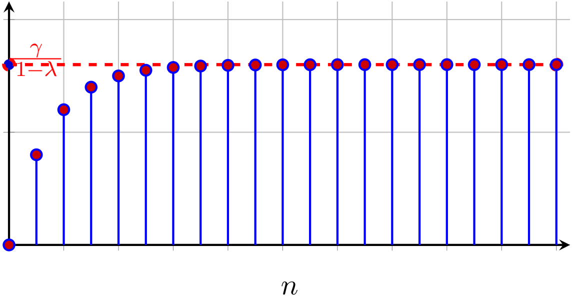
Evaluate Steady State Response

Assuming...

$$y[0] = 0, 0 < \lambda < 1, y[n] = \lambda y[n-1] + \gamma u[n-1], u[n] = 1,$$

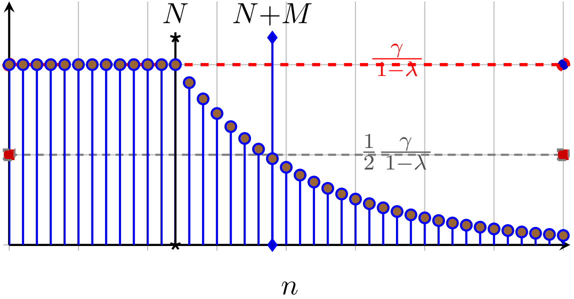
we already have one relationship!

$$y[\infty] = \frac{\gamma}{1-\lambda} \Rightarrow \gamma = y[\infty](1-\lambda).$$



Evaluate Decay

Let $u[n] = 1 \; \forall n \leq N$. How many steps does it take to decay halfway?



From Property (III) Time Invariance of ZIR:

$$\frac{\gamma}{1-\lambda}\lambda^M = 0.5\frac{\gamma}{1-\lambda}$$

$$\Rightarrow \lambda = 0.5^{1/M}.$$

$$\Rightarrow \gamma = y[\infty](1-0.5^{1/M})$$

Closing Thoughts

How can we generate the previous two plots?

- We get to pick which controller we use (and set the gains)!
- We can find an expression for λ which will (probably) be a function of the gain(s) of our controller.
- We can pick gains such that we truly reach zero steady state error.