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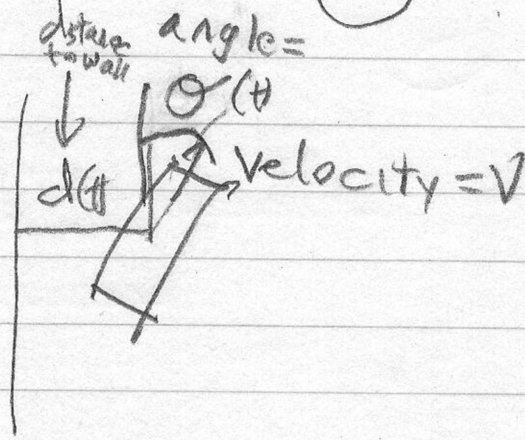


Wall Follower

$$\lim_{\Delta t \to 0} \frac{d[n]-d[n-1]}{\Delta t} = \frac{d}{dt} d(t) \simeq V\theta(t)$$

$$\frac{d}{dt}\theta(t) \simeq \gamma \underbrace{c(t)}_{\text{commanded rotation rate}}$$

Prop. Feedback $c(t) = K_p(\underbrace{d_d(t)}_{\text{desired wall distance}} - d(t))$

In D.T.

$$\underbrace{\begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix}}_{x[n]} = \underbrace{\begin{bmatrix} 1 & \Delta T V \\ \Delta T \gamma K_p & 1 \end{bmatrix}}_{A} \begin{bmatrix} d[n-1] \\ \theta[n-1] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \Delta T \gamma K_p \end{bmatrix}}_{B} \underbrace{d_d[n]}_{u}$$

$$x[n] = A^n x[n-1] + \sum_{m=0}^{n-1} A^{n-m-1} B u[m]$$

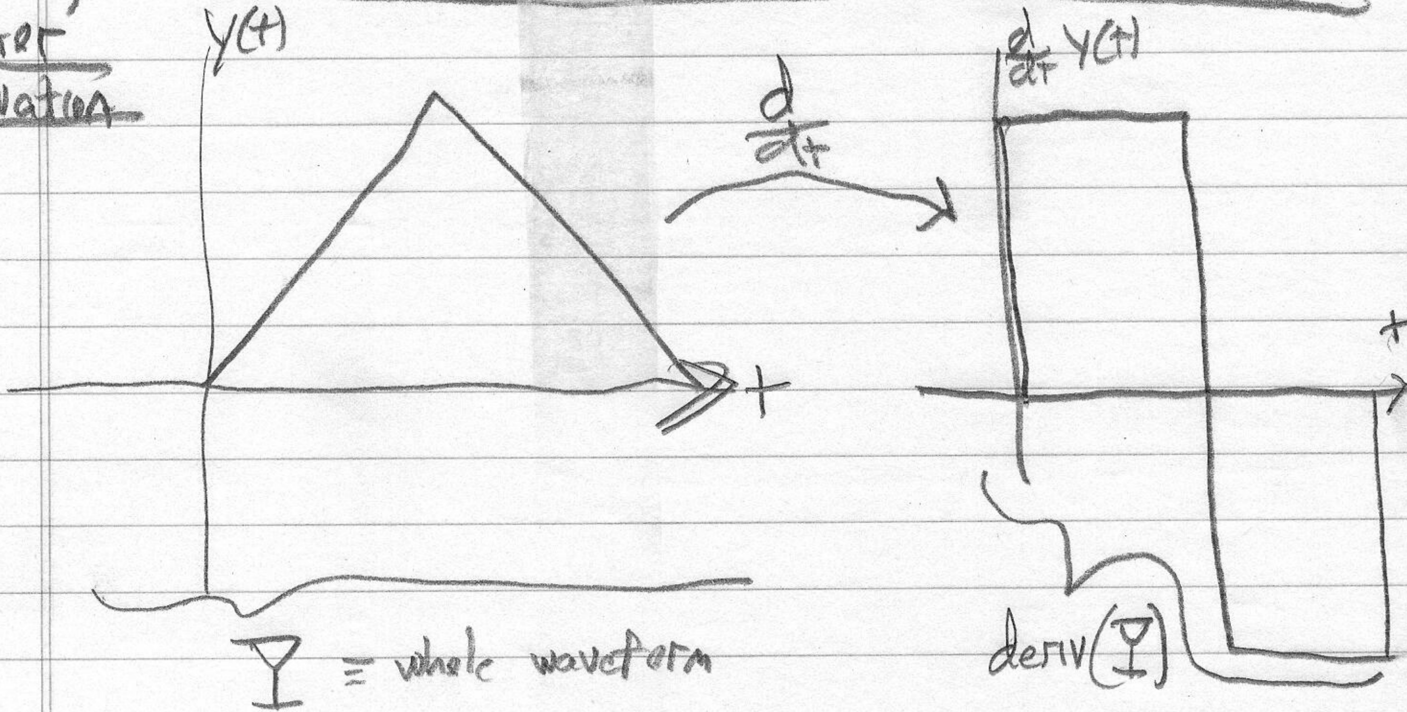
In C.T.

$$\frac{d}{dt} \underbrace{\begin{bmatrix} d(t) \\ \theta(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 0 & V \\ \gamma K_p & 0 \end{bmatrix}}_{A} \begin{bmatrix} d(t) \\ \theta(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \gamma K_p \end{bmatrix}}_{B} \underbrace{d_d(t)}_{u(t)}$$

$$x(t) = e^{At} x(0) + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}_{\text{convolution integral}}$$

The Transfer Function Approach

"Operator" Derivation



- $\frac{d}{dt}$ is an operator mapping $\mathbf{Y}$ to its deriv
- Use 's' to denote this operator

So

$$\frac{d}{dt}\, d(t) = V\, \theta(t) \Rightarrow s D = V \Phi$$

Some identities

$$s^2 \mathbf{Y} = s\left(s \mathbf{Y}\right) \Leftrightarrow \frac{d}{dt}\left(\frac{d}{dt} y(t)\right) = \frac{d^2}{dt^2} y(t)$$

scalars

$$(\beta s + \alpha)\, \mathbf{Y} \Leftrightarrow \beta \frac{d}{dt} y(t) + \alpha y(t)$$

$$(s+\alpha_1)(s+\alpha_2)\mathbf{Y} = \left(s^2 + (\alpha_1+\alpha_2)s + \alpha_1\alpha_2\right)\mathbf{Y}$$

$$\Rightarrow \frac{d^2}{dt^2} y(t) + (\alpha_1 + \alpha_2)\frac{d}{dt} y(t) + \alpha_1\alpha_2 y(t)$$

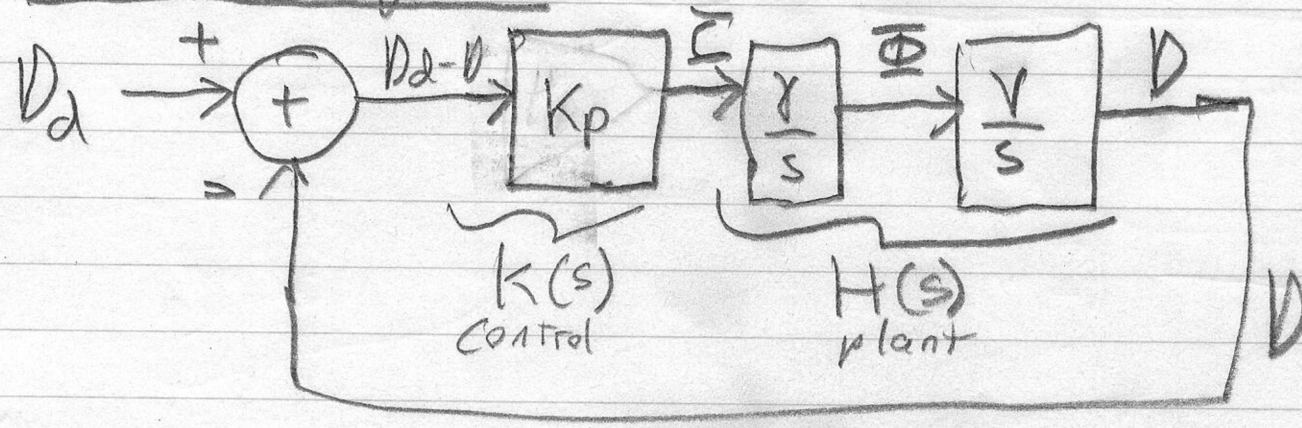
Using "S" for Line Follower

$$s D = V \Phi \qquad D = \frac{V}{s}\Phi \qquad d(t) = \int V\phi(t)$$

$$s \Phi = \gamma \Sigma \qquad \Phi = \frac{\gamma}{s}\Sigma \qquad \phi(t) = \int \gamma\sigma$$

$$\Sigma = K_p(D_d - D) \qquad \Sigma = K_p(D_d - D)$$

Block Diagram



$$D = \underbrace{K_p(D_d - D)}^{\Sigma}\left(\frac{\gamma}{s}\frac{V}{s}\right)$$

$$\left(D + D\left(K_p\frac{\gamma}{s}\frac{V}{s}\right)\right) = K_p\frac{\gamma V}{s^2}D_d$$

$$\left(1 + K_p\frac{\gamma}{s}\frac{V}{s}\right)D = K_p\frac{\gamma V}{s^2}D_d$$

$$D = \frac{s^2 \quad K_p\frac{\gamma V}{s^2}}{s^2 \quad 1 + K_p\frac{\gamma V}{s^2}}D_d$$

is this justified

is $\frac{s^2}{s^2}$ really the identity.

$$D = \underbrace{\frac{K_p V \gamma}{s^2 + K_p V \gamma}}_{G(s)} D_d$$

Cross Multiply ⇒ $\left(s^2 + K_p \gamma V\right) D = K_p V \gamma D_d$

I/O Diff. Eqn ⇒ $\left(\frac{d^2}{dt^2} + K_p \gamma V\right) d(t) = K_p V \gamma \, d_d(t)$

We can use operator calculus to determine I/O differential eqn

Determining Natural Frequencies

Zero Input Response $\quad d_d(t) = 0 \quad d(0) \neq 0$

$$\frac{d^2}{dt^2} d(t) + K_p \gamma V d(t) = 0$$

Suppose $\quad d(t) = Ae^{\lambda t} \Rightarrow \frac{d}{dt} d(t) = A\lambda e^{\lambda t}$

$$\frac{d^2}{dt^2} = A\lambda^2 e^{\lambda t}$$

⇒

$$\cancel{A}\lambda^2 \cancel{e^{\lambda t}} + K_p \gamma V \cancel{A e^{\lambda t}} = 0$$

$$\lambda^2 = -K_p \gamma V \Rightarrow \lambda = \pm j\sqrt{K_p \gamma V}$$

Or generally

$$d(t) = A_1 e^{\overset{j\sqrt{K_p\gamma V}}{\lambda_1} t} + A_2 e^{\overset{-j\sqrt{K_p\gamma V}}{\lambda_2} t}$$

If $d(0)$, $\sigma(0)$ are given, then

$$\frac{d}{dt} d(t) = V\sigma(t) \Rightarrow \left.\left(\frac{d}{dt} d(t)\right)\right|_{t=0} = V\sigma(0)\Big|_{t=0}$$

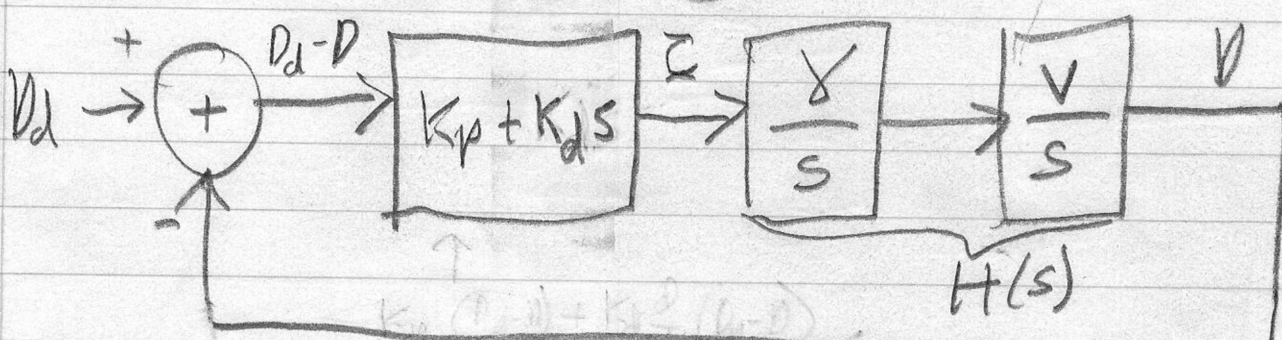
Given $d(0)$, $\left.\frac{d}{dt} d(t)\right|_{t=0}$ can find A_1, A_2

Stable?

Does the ZIR → 0 as $t \to \infty$

$$e^{j\sqrt{K_p \gamma V}\,t} = \underbrace{\cos\left(\sqrt{K_p \gamma V}\right)t + j\sin\left(\sqrt{K_p \gamma V}\right)t}_{\lim_{t\to\infty} \neq 0}$$

Control with $\overbrace{\text{Prop. FdBk}}^{K_p}$ is not stable!



P.D. Control $\quad c(t) = K_p(d_d(t) - d(t)) + K_d \frac{d}{dt}(d_d(t) - d(t))$

$$\Rightarrow K_p(D_d - D) + K_d s(D_d - D)$$

$$D = K(s)\,H(s)\,(D_d - D)$$

$$\left(1 + K(s)H(s)\right) D = K(s)H(s)\,D_d$$

Black's Formula : $\quad D = \dfrac{K(s)H(s)}{\underbrace{1 + K(s)H(s)}_{G(s)}} D_d$

$$K(s)H(s) = \frac{(K_p + K_d s)\gamma V}{s^2} = \frac{n(s)}{d(s)}$$

numerator poly in s

denom poly in s

$$D = \frac{\frac{n(s)}{d(s)}}{1 + \frac{n(s)}{d(s)}} D_d = \frac{n(s)}{d(s) + n(s)} D_d$$

$$D = \frac{(K_p + K_d s)\gamma V}{s^2 + (K_p + K_d s)\gamma V} V_d$$

Unnecessary

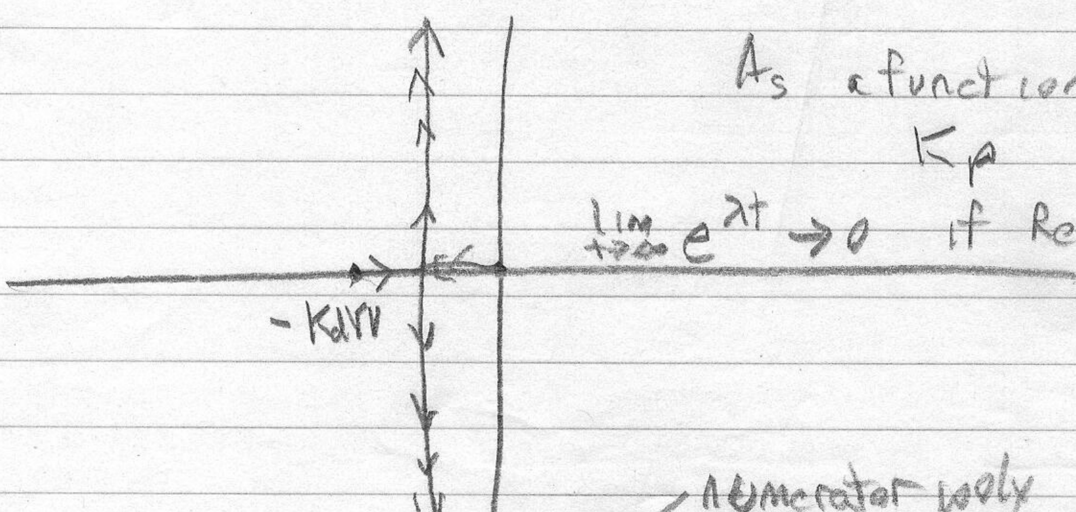
$$(s^2 + K_d \gamma V s + K_p \gamma V) D = (K_p + K_d s) V_d$$

$$\left(\frac{d^2}{dt^2} + K_d \gamma V \frac{d}{dt} + K_p \gamma V\right) d(t) = 0 \qquad \underline{ZIR}_{V_d = 0}$$

Assume $e^{\lambda t}$ solutions

$$\underbrace{(\lambda^2 + K_d \gamma V \lambda + K_p \gamma V)}_{\text{roots are nat freqs!}} e^{\lambda t} = 0$$

$$\lambda_1 \lambda_2 = -\frac{K_d \gamma V}{2} \pm \sqrt{\frac{(K_d \gamma V)^2}{4} - K_p \gamma V}$$



As a function of K_p

$\lim_{t \to \infty} e^{\lambda t} \to 0$ if $Re(\lambda) < 0$

If $G(s) = \frac{n_g(s)}{d_g(s)}$ ← numerator poly in s, ← denom poly in s

Nat freqs are roots of $d_g(s)$!

Solns of $d_g(s) = 0$