Higher Order Models

So Far . . .

- Model dynamical input/output effects by *first order models*:

$$y[n] = \lambda y[n-1] + \gamma u[n-1]$$

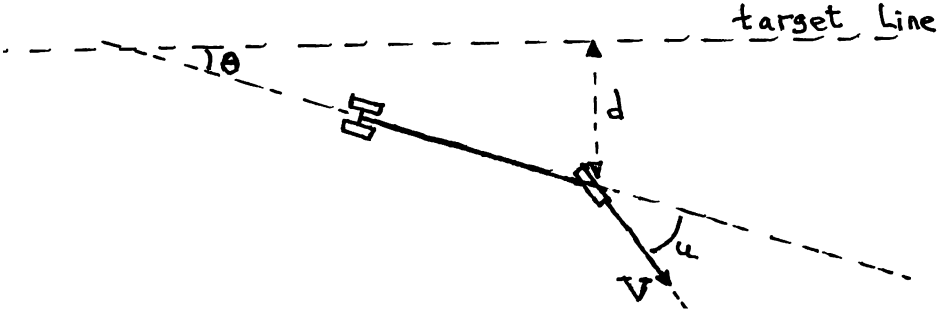
  ($u[n]$ is n-th input sample, $y[n]$ is n-th output sample). These models are
  - ✓ linear
  - ✓ time-invariant
  - ✓ strictly causal
- Assess *stability* by comparing $|\lambda|$ to 1.
- Enjoy superposition laws and semi-explicit formulae for the output, like

$$y[n] = \lambda^n y[0] + \gamma \sum_{k=0}^{n-1} \lambda^{n-k-1} u[k].$$

Are first order models enough for what we need in 6.3100 ?

A Front-Wheel-Drive Tricycle

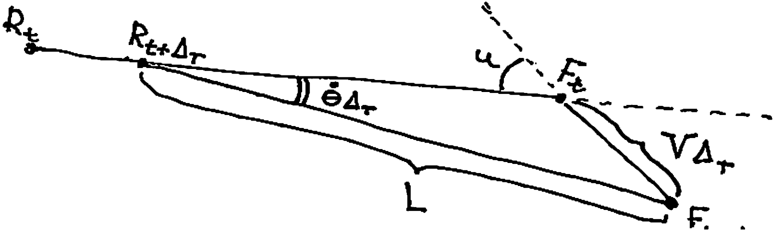
A Front-Wheel-Drive Tricycle



Notation

- V : front wheel forward speed (constant, in *m/s*)
- L : front wheel to rear axle center distance (constant, in *m*)
- θ : heading angle (signed, in *rad*, output signal)
- d : distance to the target line (signed, in *m*, controlled output)
- u : front wheel to robot axis angle (signed, in *rad*, control input)

Differential equations for front-wheel-drive tricycle



- Law of Sines: $\frac{V\Delta_T}{\sin(\dot{\theta}\Delta_T)} = \frac{L}{\sin(u)}$, hence $\dot{\theta} = \frac{V}{L}\sin(u)$
- Change of d: $\dot{d} = V\sin(\theta + u)$

Assuming $|u|$ and $|\theta|$ are small (i.e., $\sin(\theta + u) \approx \theta + u$, $\sin(u) \approx u$)

$$\dot{d} = V\sin(\theta + u) \approx V(\theta + u),$$
$$\dot{\theta} = \frac{V}{L}\sin(u) \approx \frac{V}{L}u.$$

Exact ZOH Difference Equations for Front-Wheel-Drive Tricycle

- $u[n]$: constant value of $u(t)$ on $[n\Delta_T, n\Delta_T + \Delta_T)$
- $\theta[n] = \theta(n\Delta_T)$, $d[n] = d(n\Delta_T)$: samples of θ and d

Since $\dot{\theta}(t) = \frac{V}{L}u(t) = \frac{V}{L}u[n]$ for t in $[n\Delta_T, n\Delta_T + \Delta_T)$,

$$\theta(n\Delta_T + \tau) = \theta[n] + \frac{V}{L}u[n]\tau \quad \text{for} \quad 0 \le \tau \le \Delta_T.$$

In particular, $\theta[n+1] = \theta[n] + \frac{V}{L}\Delta_T u[n]$.

Since $\dot{d}(t) = V(\theta(t) + u(t)) = V(\theta[n] + u[n] + \frac{V}{L}u[n]\tau)$ for $t = \Delta_T + \tau$, $0 \le \tau < \delta_T$,

$$d(n\Delta_T + \tau) = d[n] + V(\theta[n] + u[n])\tau + \frac{V^2}{2L}u[n]\tau^2 \quad \text{for} \quad 0 \le \tau \le \Delta_T.$$

In particular, $d[n+1] = d[n] + V\Delta_T\theta[n] + (V\Delta_T + \frac{V^2\Delta_T^2}{2L})u[n]$.

Approximate ZOH Difference Equations for Front-Wheel-Drive Tricycle

- $u[n]$: constant value of $u(t)$ on $[n\Delta_T, n\Delta_T + \Delta_T)$
- $\theta[n] = \theta(n\Delta_T)$, $d[n] = d(n\Delta_T)$: samples of θ and d

With Δ_T sufficiently small (more accurately, when $V\Delta_T/L \ll 1$),

$$\dot{d} = V(\theta + u), \quad \dot{\theta} = \frac{V}{L}u$$

converts to

$$d[n+1] = d[n] + V\Delta_T(\theta[n] + u[n]) + \frac{V^2\Delta_T^2}{2L}u[n] \approx d[n] + V\Delta_T(\theta[n] + u[n]),$$
$$\theta[n+1] = \theta[n] + \frac{V\Delta_T}{L}u[n].$$

One can argue that the missing $\frac{V^2\Delta_T^2}{2L}u[n]$ term in the first equation is not important, as everything is known approximately anyway.

Matrix form

Define the state

$$\boldsymbol{x}[n] := \begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix}.$$

Then the small-angle approximation

$$d[n+1] = d[n] + V\Delta_T(\theta[n] + u[n]) + \frac{V^2\Delta_T^2}{2L}u[n] \approx d[n] + V\Delta_T(\theta[n] + u[n]),$$

$$\theta[n+1] = \theta[n] + \frac{V\Delta_T}{L}u[n].$$

can be converted to

$$\boldsymbol{x}[n+1] = A\,\boldsymbol{x}[n] + B\,u[n], \qquad A = \begin{bmatrix} 1 & V\Delta_T \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} V\Delta_T(1 + \frac{V\Delta_T}{2L}) \\ \frac{V\Delta_T}{L} \end{bmatrix}.$$

Sanity Checks ?

$$\boldsymbol{x}[n] := \begin{bmatrix} d[n] \\ \theta[n] \end{bmatrix}, \qquad x[n+1] = Ax[n] + Bu[n].$$

Scalars hiding within the state

- To recover $d[n]$ as $Cx[n]$, which C should I use?
- To recover $\theta[n]$ as $C_\theta x[n]$, which C_θ should I use?

The meaning of proportional feedback

We expect the use of proportional feedback $u[n] = K_p(d_{des}[n] - d[n])$ to result in a similar equation

$$x[n+1] = A_{cl}x[n] + B_{cl}d_{des}[n],$$

where A_{cl} and B_{cl} are defined in terms of A, B, and K_p. What is the expression for A_{cl} and B_{cl}?

General solution (scalar vs. vector)

Scalar recurrence

If $y[n] = \lambda\, y[n-1] + \gamma\, u[n-1]$, then

$$y[n] = \lambda^n y[0] + \sum_{m=0}^{n-1} \lambda^{n-1-m} \gamma\, u[m].$$

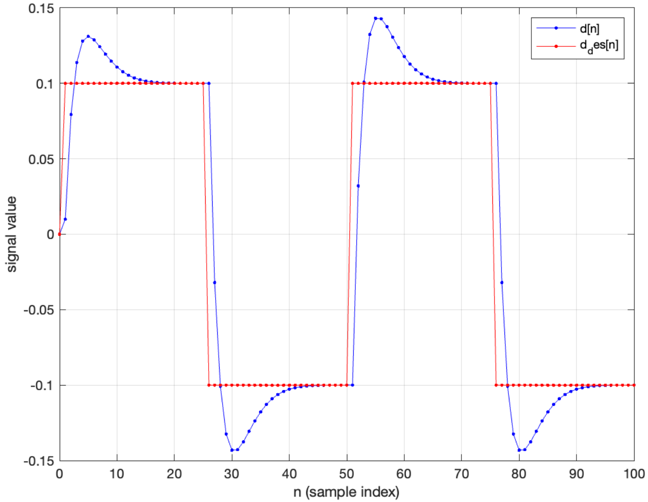
Vector recurrence

If $\boldsymbol{x}[n] = A\boldsymbol{x}[n-1] + B\, u[n-1]$, then

$$\boldsymbol{x}[n] = A^n \boldsymbol{x}[0] + \sum_{m=0}^{n-1} A^{n-1-m} B\, u[m].$$

Matrices generally do *not* commute, so the order matters.

Front Wheel Drive Transients ($L = V = 1$, $\Delta_T = 0.2$, $K_p = 3$)



Which Features of A_{cl} determine the transients, qualitatively?

- Do the transients settle?
- Do the transients oscillate?
- What is the growth/decay rate of the transients?

Which properties of matrices do we traditionally pay attention to?

Dependence of components of A^n on $n = 0, 1, 2, \ldots$

- if N-by-N real matrix A has N **different** eigenvalues $\lambda_1, \ldots, \lambda_N$ (some of which could be complex) then each component of A^n will be a sum of terms λ_k^n scaled with constant (possibly complex) coefficients
- if an eigenvalue λ of A is complex, i.e., $\lambda = re^{j\phi}$ for some real $r > 0$ and $\phi \in (-\pi, \pi)\backslash\{0\}$, then $\bar{\lambda} = re^{-j\phi}$ is also an eigenvalue of A, and the corresponding λ^n and $\bar{\lambda}^n$ terms in A^n can be replaced by the terms $r^n \cos(n\phi)$ and $r^n \sin(n\phi)$
- if an eigenvalue $\lambda \neq 0$ of A is repeated $m > 1$ times, in addition to the λ^n term, one may need to add terms $n\lambda^n$, $n^2\lambda^n$,..., $n^{m-1}\lambda^n$ to form the components of A^n

Asymptotic behavior of A^n

- $A^n \to 0$ as $n \to +\infty$ if and only if all eigenvalues of A have absolute value less than 1
- a component of A^n grows exponentially as $n \to +\infty$ if and only if an eigenvalue of A has absolute value greater than 1
- components of A^n have oscillations (possibly hidden behind a stronger real exponent) when A has non-real eigenvalues

Diagonalizable matrices

A **square** real N-by-N matrix $A \in \mathbb{R}^{N \times N}$ is called **diagonalizable** when

$$A = S\Lambda S^{-1}, \quad \text{where} \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{bmatrix}$$

which implies

$$A^n = S\Lambda S^{-1} S\Lambda S^{-1} \ldots S\Lambda S^{-1} = S\Lambda^n S^{-1} = S \begin{bmatrix} \lambda_1^n & 0 & \ldots & 0 \\ 0 & \lambda_2^n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N^n \end{bmatrix} S^{-1}$$

Diagonalizable matrices (continued)

$$A = S\Lambda S^{-1}, \quad \text{where} \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{bmatrix}$$

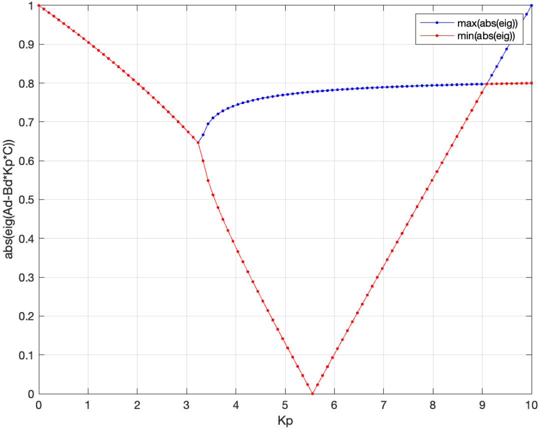
means

- λ_k are **eigenvalues** of A (possibly repeated)
- columns of S are **eigenvectors** of A, **linearly independent**
- even though A is real, Λ and S may have to be complex

Two simple **sufficient** criteria of diagonalizability of A:

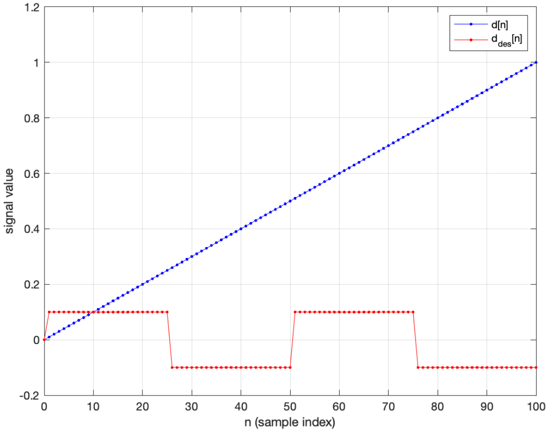
- when N-by-N matrix A has N **different** eigenvalues
- when A and its transpose A^T commute, i.e., $AA^T = A^TA$ (e.g., symmetric A)

Front Wheel Drive Tricycle Eigenvalues ($L = V = 1$, $\Delta_T = 0.2$)



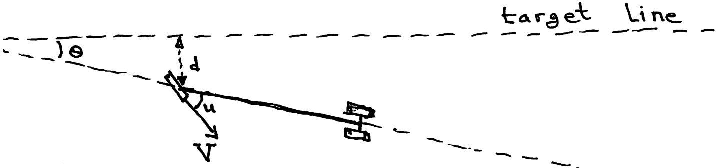
Here we see stability for $0 < K_p < 10$, and complex eigenvalues for $0 < K_p < K_p^0 \approx 3.2$

Front Wheel Drive Transients ($L = V = 1$, $\Delta_T = 0.2$, $K_p = 0$)



You can also see how the linear model breaks down here!

A Rear-Wheel-Drive Tricycle



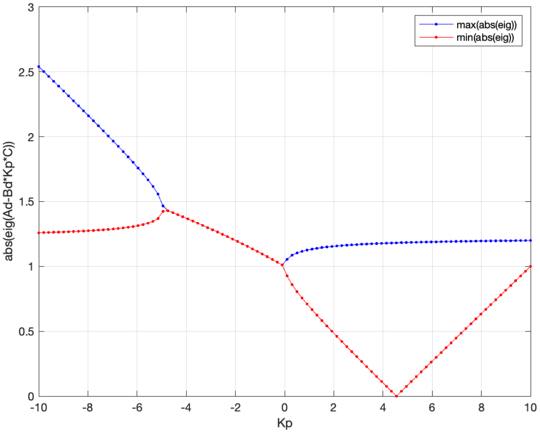
Differential equation model

$$\dot{d} = V(\theta + u) \approx V(\theta + u),$$
$$\dot{\theta} = -\frac{V}{L}\sin(u) \approx -\frac{V}{L}u$$

converts to

$$d[n+1] = d[n] + V\Delta_T(\theta[n] + u[n]) - \frac{V^2\Delta_T^2}{2L}u[n],$$
$$\theta[n+1] = \theta[n] - \frac{V\Delta_T}{L}u[n].$$

Rear Wheel Drive Tricycle Eigenvalues ($L = V = 1$, $\Delta_T = 0.2$)



There is no way to stabilize with a proportional feedback!

Quiz du Jour

Control system state $y = y[n]$ is updated according to the law

$$y[n] = Ay[n-1], \qquad \text{with} \quad y[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(Naturally, A is a real matrix.) One of the eigenvalues of A is known to be $\lambda = 0.5 + 0.5j$ (where $j = \sqrt{-1}$).

(a) What is the **other** eigenvalue one can be sure A also has?

(b) Is it guaranteed that A is diagonalizable?

(c) Does the length of $y[n]$ decay or grow as $n \to +\infty$?

(d) Do the components of $y[n]$ oscillate or not as $n \to +\infty$?