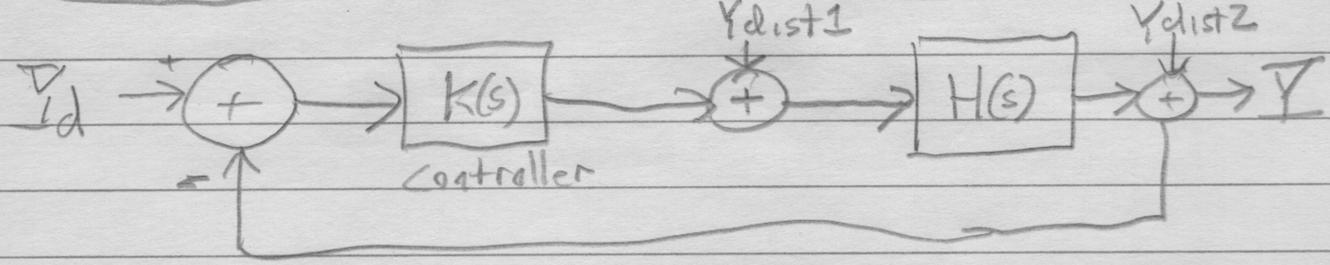
3/16/26 6.310

Transfer function in Block Diagrams



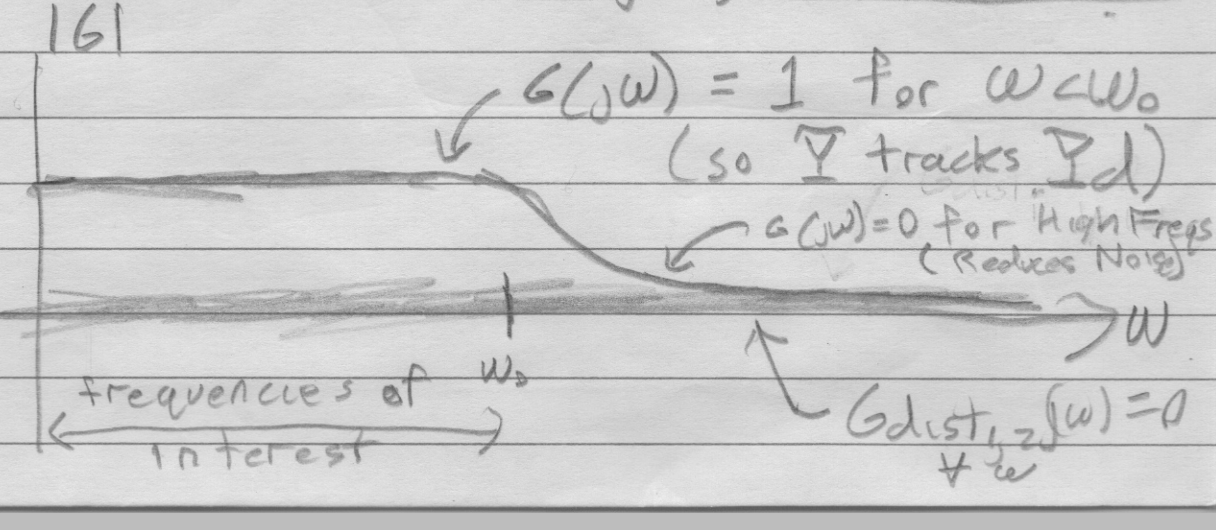
$$Y = \underbrace{\frac{K(s)H(s)}{1+K(s)H(s)}}_{G(s)} Y_d + \underbrace{\frac{H(s)}{1+K(s)H(s)}}_{G_{dist1}(s)} Y_{dist1} + \underbrace{\frac{H(s)}{1+K(s)H(s)}}_{G_{dist2}(s)} Y_{dist2}$$

In sinusoidal steady state

- System Must Be Stable (or no SSS)
  - Nat. Freqs: λ_i s.t. $1+K(\lambda_i)H(\lambda_i)=0$
  - $Re(\lambda_i) < 0$

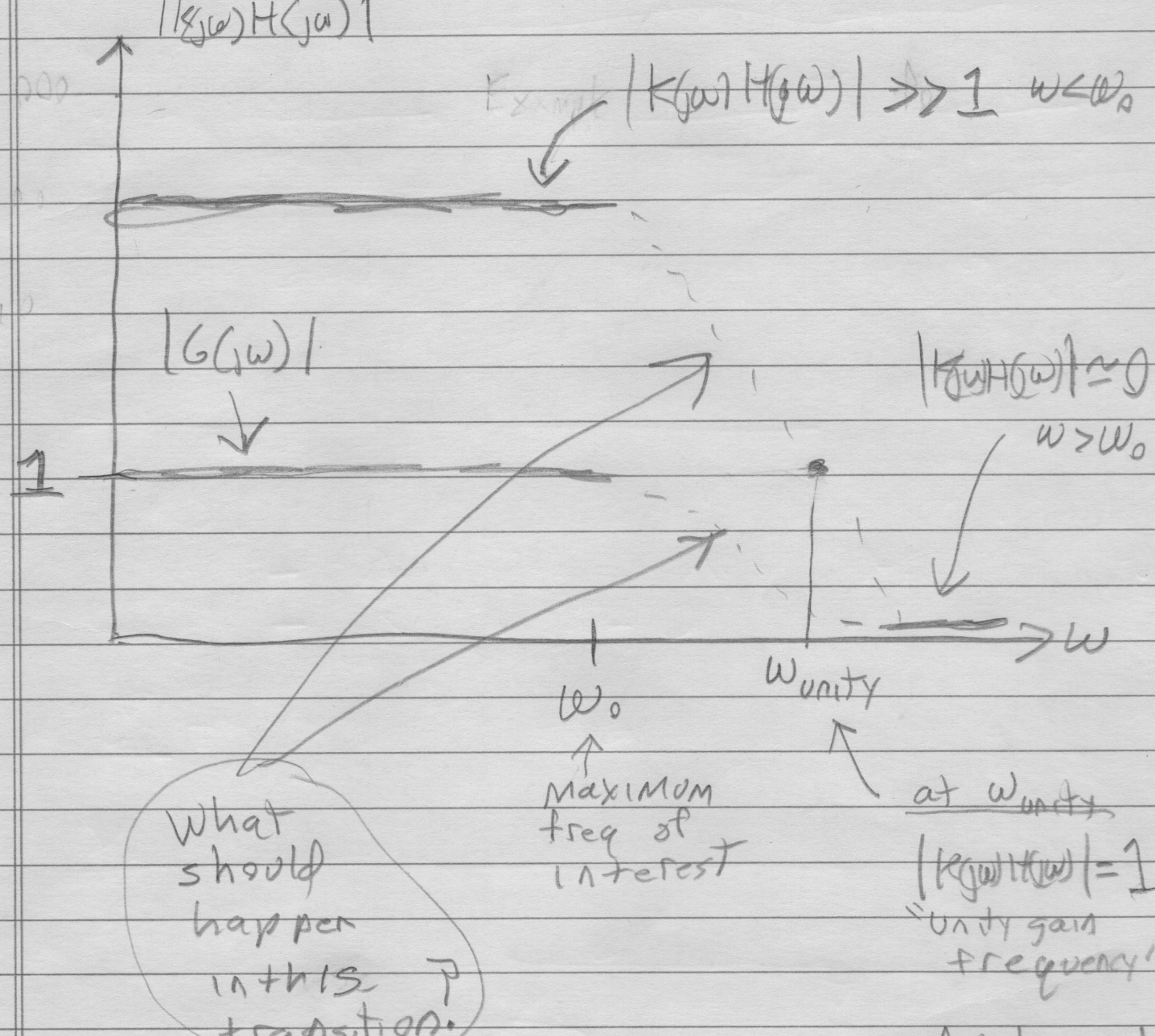
$$Y = G(j\omega) Y_d + G_{dist}(j\omega) Y_{dist1} + G_{dist2}(j\omega) Y_{dist2}$$

Most Desirable $G(j\omega)$, $G_{dist1}(j\omega)$, $G_{dist2}(j\omega)$



$$G(j\omega) = \frac{K(j\omega)H(j\omega)}{1 + K(j\omega)H(j\omega)} \simeq 1 \quad \text{if } |K(j\omega)H(j\omega)| \gg 1$$

$$G(j\omega) \simeq 0 \quad \text{if } |K(j\omega)H(j\omega)| \ll 1$$



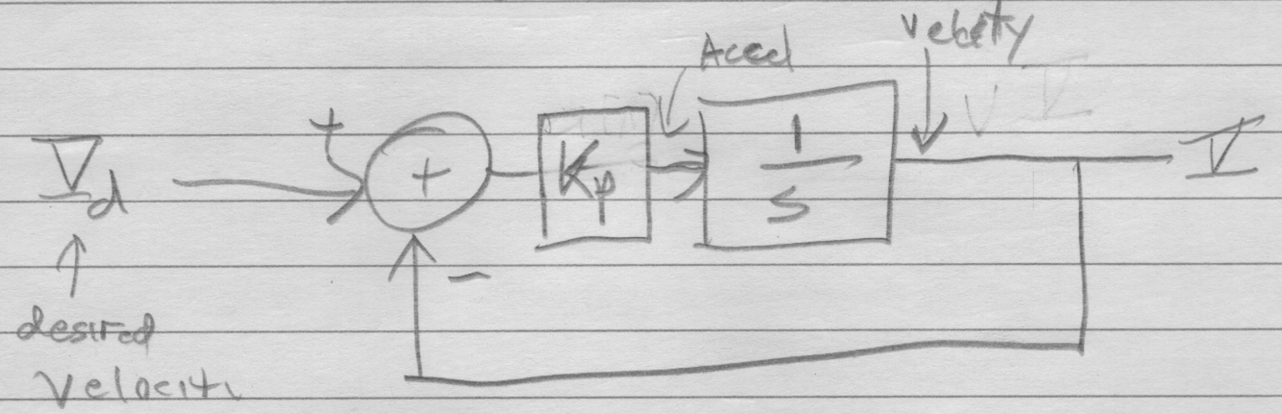
Disturbance? if $K(j\omega)H(j\omega) \gg 1$ then

Disturbances Rejected!

$$G_{dist1}(j\omega) \simeq \frac{H(j\omega)}{1+K(j\omega)H(j\omega)} \simeq \frac{1}{K(j\omega)} \simeq 0 \qquad G_{dist2}(j\omega) \simeq 0$$

3

Example 1 Velocity Control with proportional feedback



$$G(s) = \frac{K_p/s}{1 + K_p/s} = \frac{K_p}{s + K_p}$$

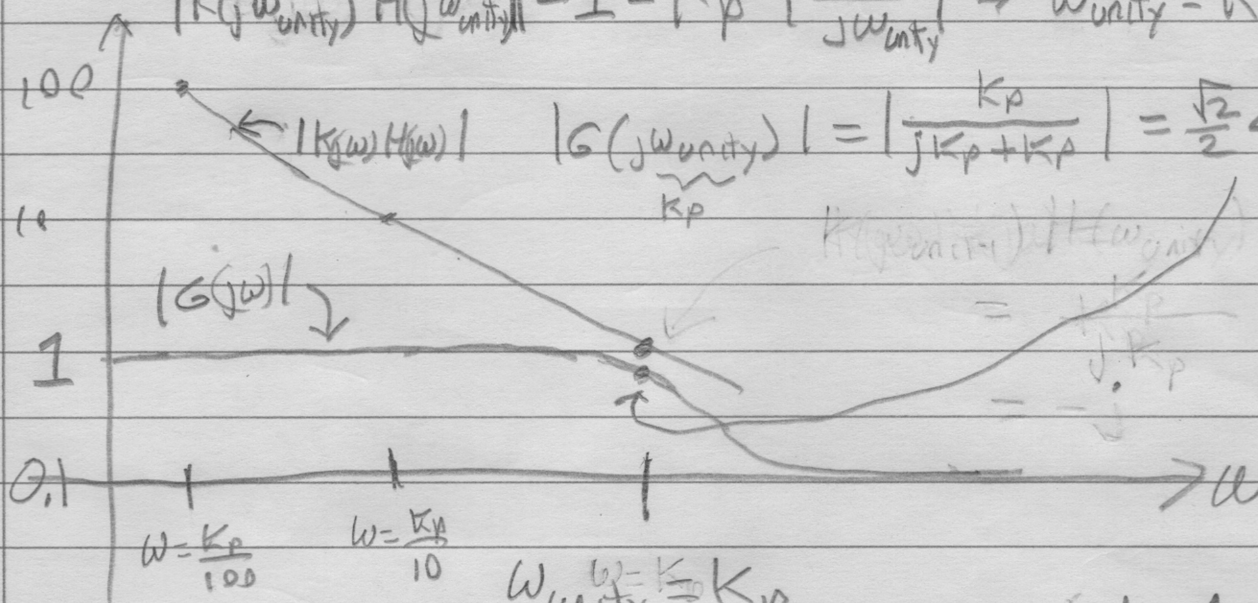
Natural Freqs of $G(s)$: $\lambda = -K_p$ $Re(\lambda) < 0$ Stable!

$$G(j\omega) = \frac{K_p/j\omega}{1 + K_p/j\omega} = \frac{K_p}{j\omega + K_p}$$

$$|K(j\omega)H(j\omega)| = |K_p/j\omega| = \frac{K_p}{\omega}$$

$$|K(j\omega_{unity})H(j\omega_{unity})| = 1 = K_p\left|\frac{1}{j\omega_{unity}}\right| \Rightarrow \omega_{unity} = K_p!$$

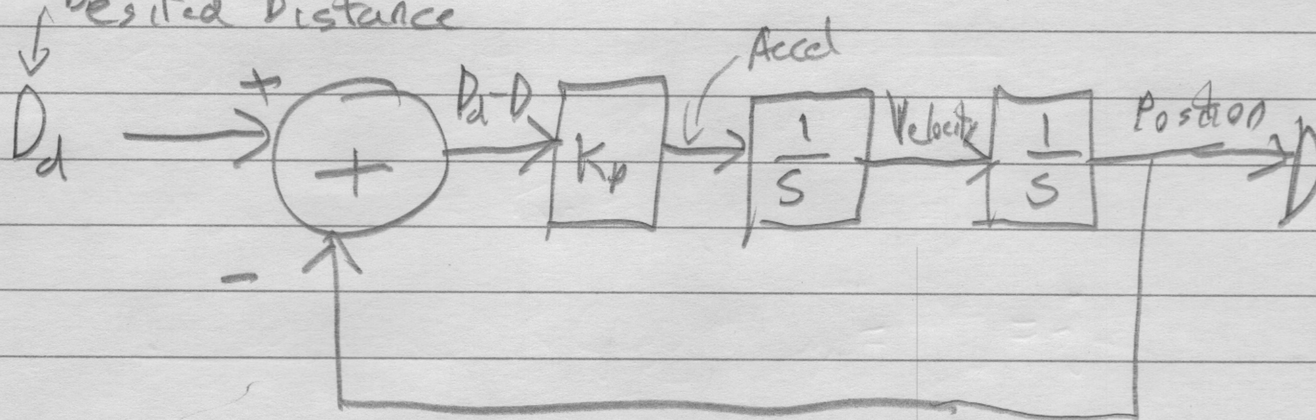
$$|G(j\underbrace{\omega_{unity}}_{K_p})| = \left|\frac{K_p}{jK_p + K_p}\right| = \frac{\sqrt{2}}{2} < 1$$



Magnitude 1

$$K(\omega_{unity})H(j\omega_{unity}) = -j \leftarrow \angle -90^\circ$$

Example 2 Proportional Control of Position

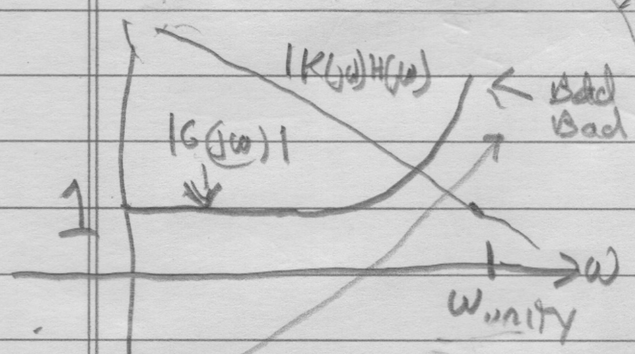


$$K(s)\,H(s) = K_p \frac{1}{s^2} = \frac{K_p}{s^2}$$

$$G(s) = \frac{K(s)H(s)}{1 + K(s)H(s)} = \frac{K_p/s^2}{1 + K_p/s^2}$$

$$= \frac{K_p}{s^2 + K_p}$$

Frequency Response



Nat Freqs

λ s.t. $\lambda^2 + K_p = 0$

$\lambda = \pm\sqrt{-K_p} = \underbrace{\pm j\sqrt{K_p}}_{\text{purely imaginary}}$

$Re(\lambda) = 0$

Not < 0

Not stable!

$$K(j\omega)H(j\omega) = \frac{K_p}{(j\omega)^2} = \frac{K_p}{-\omega^2} \Rightarrow \omega_{unity} = \sqrt{K_p}$$

Bad Bad

$$K(j\omega_{unity})H(j\omega_{unity}) = \frac{K_p}{-(\sqrt{K_p})^2} = -1$$

$$\infty = \frac{-1}{1 + -1} = G(j\omega_{unity}) = K(j\omega_{unity})H(j\omega_{unity}) \Big/ \left(1 + K(j\omega_{unity})H(j\omega_{unity})\right)$$

For velocity control example

$$K(s)H(s)\Big|_{s=j\omega} = \frac{K_p}{s}\Big|_{s=j\omega} = \frac{K_p}{j\omega}$$

$$\omega_{unity} = K_p$$

At ω_{unity} $\quad K(j\omega_{unity})H(j\omega_{unity}) = \frac{K_p}{jK_p} = -j$

magnitude $= 1$

angle (or phase) $= -90^\circ$

$$|G(j\omega_{unity})| = \left|\frac{-j}{1+(-j)}\right| = \frac{1}{\sqrt{2}}$$

For position control example

$$K(s)H(s)\Big|_{s=j\omega} = \frac{K_p}{s^2}\Big|_{s=j\omega} = -\frac{K_p}{\omega^2}$$

$$\omega_{unity} = \sqrt{K_p}$$

At ω_{unity} $\quad K(j\omega_{unity})H(j\omega_{unity}) = -1$

magnitude $= 1$

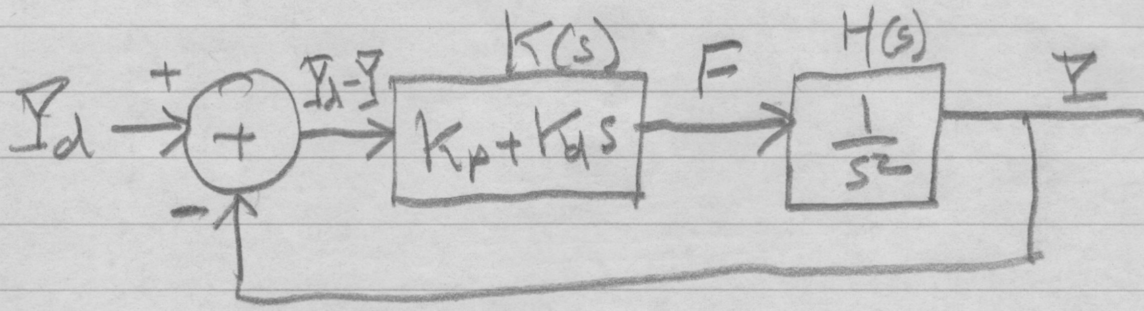
angle (or phase) $= -180^\circ$

$$G(j\omega_{unity}) = \frac{K(j\omega_{unity})H(j\omega_{unity})}{1+K(j\omega_{unity})H(j\omega_{unity})}$$

Keep $K(j\omega_{unity})H(j\omega_{unity})$ away from -1!

$$|G(\omega_{unity})| = -1/(1+(-1)) = \infty\,!$$

Use a PD controller



$$K(j\omega)H(j\omega) = \frac{K_p + K_d j\omega}{(j\omega)^2} = -\frac{K_p + j\omega K_d}{\omega^2}$$

$$= -\frac{K_p}{\omega^2} - j\frac{K_d}{\omega}$$

$$\angle K(j\omega)H(j\omega) \xrightarrow[\omega \to 0]{} \angle -\frac{K_p}{\omega^2} = -180^\circ$$

$$\angle K(j\omega)H(j\omega) \xrightarrow[\omega \to \infty]{} \angle -j\frac{K_d}{\omega} = -90^\circ$$