6.3100 April 22, 2026 Lecture:
State Estimation: an Example

Last Lecture

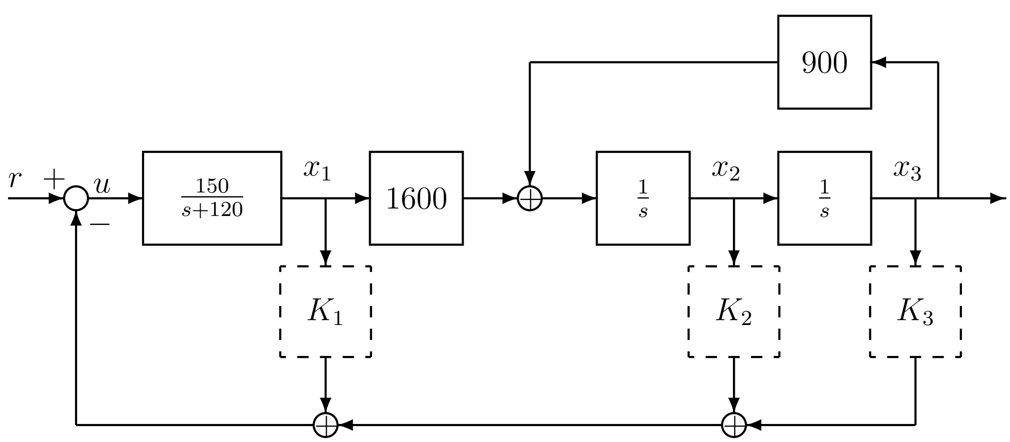
- Guaranteed robustness of LQR to relative modeling errors.
- Other robustness guarantees the can be addressed by LQR.

Outline for Today

- Example: estimate the middle state of a double integrator chain

What if Not All States Are Available for Measurement?

The formulation of LQR optimization assumes that all components of the model's state are available for measurement For example, the full state feedback block diagram for the maglev system shown below:

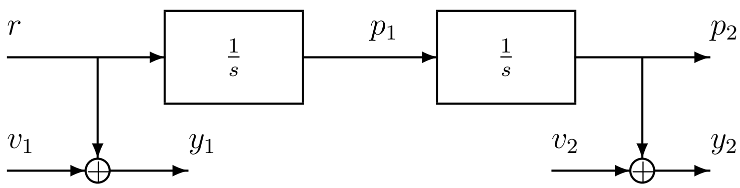


assumes that states x_1, x_2, x_3 are available for control action computation. In practice, only a few state components are measured: two (x_1 and x_3) in our maglev experiment, only one (angular position) in our propeller arm control setup. Worse, all measurements involve errors, typically referred to as "noises", which means that trying to recover non-measured states from the measured ones by (numerical) differentiation will incur significant amplification of noise.

It turns out that there is a systematic way of recovering unmeasured states from the measured ones which does not lead to any unnecessary noise amplification. This is done by feeding the control command and all measurements as inputs into a specially designed state space model, the states of which can then serve as *estimates* of the true states of the original system model. Moreover, this "estimator" system can be optimized via the very same LQR algorithm we have discussed, just applied to a slightly weird state space model (actually, to optimize the estimator, LQR is applied to the model that is "dual" to the original one). The estimates usually turn out to be so good, that, for state feedback, it is usually better to use estimates of the measured states, instead of the actual measurements! Our objective for the next few lectures is to study the format of state estimation, the type of modeling that precedes its optimization, the use of LQR in optimizing the coefficients of the estimator, as well as limitations of the whole paradigm.

Example: Recovering the Middle Signal in a Chain of Integrators

Consider the (rather common) scenario when both the input r and the output p_2 of a chain of two pure integrators are measured (with noises v_1 and v_2, respectively), while the output p_1 of the first integrator is not, as described by the block diagram

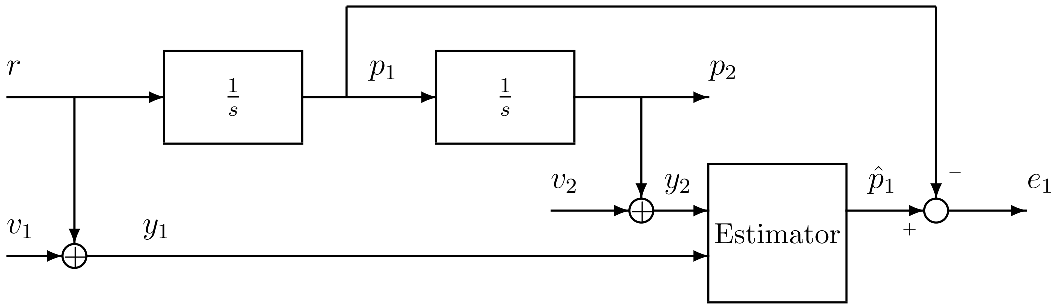


or, equivalently, equations

$$\begin{aligned}
\dot{p}_1(t) &= r(t), \\
\dot{p}_2(t) &= p_1(t), \\
y_1(t) &= r(t) + v_1(t), \\
y_2(t) &= p_2(t) + v_2(t).
\end{aligned}$$

For example, this is the case for our maglev system on the earlier block diagram, with $r = 1600x_1 + 900x_3$, $p_2 = x_3$, $p_1 = x_2$. Since both x_1 and x_3 are measured (with some noise), it is little stretch to state that r and p_2 are measured. How to estimate $p_1(t)$, as accurately as possible, from signals $y_1(t)$ and $y_2(t)$?

Let us denote the desired estimate of $p_1(t)$ by $\hat{p}_1(t)$. We are trying to design an estimator as a linear dynamical system with constant coefficients, two inputs (y_1 and y_2), and single output $\hat{p}_1$.



We will require the transfer function $H_r(s)$ from r to the resulting estimation error $e = \hat{p}_1 - p_1$ to be zero, and, for simplicity, will quantify quality of estimation by the sum of maximal absolute values of the frequency responses $H_k(j\omega)$ from each of the noises v_k to the estimation error e. (The more appropriate measure of estimation quality would be the sum of the *integrals* of squares of absolute values of the frequency responses.)

Two "direct" approaches to creating $\hat{p}_1(t)$ fail quite miserably:

Using $\hat{p}_1(t) = \dot{y}_2(t)$. Since p_1 is the derivative of p_2, and y_2 is the measured value of p_2, it is natural to try the derivative of y_2 in the role of $\hat{p}_1$. While the resulting transfer functions from r and v_1 to e are zero, the transfer function $H_2(s)$ from v_2 to e is s, hence the maximal value of $|H_2(j\omega)| = |\omega|$ over all real ω is infinity.

Using $\hat{p}_1(t)$ as an integral of $y_1(t)$ Since r is the derivative of p_1, and y_1 is the measured value of r, it is natural to try the integral of y_1 in the role of $\hat{p}_1$. While the resulting transfer functions from r and v_2 to e are zero, the transfer function $H_1(s)$ from v_1 to e is $1/s$, hence the maximal value of $|H_1(j\omega)| = 1/|\omega|$ over all real ω is infinity.

To find a better way of estimating p_1, note that the "integral of y_1" approach failed because the pure integrator is an unstable system. For example, if the differential equation for p_1 was

$$\dot{p}_1 = -p_1 + r = -p_1 + y_1 - v_1,$$

it would be much more reasonable to define $\hat{p}_1$ by $d\hat{p}_1/dt = -\hat{p}_1 + y_1$.

If the issue with the "integral of y_1" approach is instability of the equation updating p_1, we can try to find a *linear combination* $p_* = p_1 - Lp_2$ of p_1 and p_2, where L is a real parameter to be figured out, for which the differential update equation is stable. Since

$$\begin{aligned}
\dot{p}_* &= \dot{p}_1 - L\dot{p}_2 \\
&= r - Lp_1 \\
&= r - L(p_1 - Lp_2) - L^2 p_2 \\
&= -Lp_* + y_1 - L^2 y_2 - v_1 + L^2 v_2,
\end{aligned}$$

the update dynamics of p_* is stable whenever $L > 0$, and a hence a dynamic estimate $\hat{p}_*(t)$ of $p_*(t)$ can be obtained by copying the equation fpr $\dot{p}_*$ and throwing away all unknown terms in it:

$$\frac{d\hat{p}_*}{dt} = -L\hat{p}_* + y_1 - L^2 y_2,$$

which guarantees that $e_*(t) = \hat{p}_*(t) = p_*(t)$ has stable dependence on the noises:

$$\dot{e}_*(t) = -Le_*(t) + v_1 - L^2 v_2.$$

Moreover, since $p_1(t)$ cab be expressed in terms of $p_*(t)$ as in

$$\begin{aligned} p_1 &= p_* + Lp_2 \\ &= p_* + Ly_2 - Lv_2, \end{aligned}$$

it is natural to define

$$\hat{p}_1(t) = \hat{p}_*(t) + Ly_2(t),$$

which also yields the expression for the estimation error $e_1(t) = \hat{p}_1(t) - p_1(t)$:

$$e_1(t) = e_*(t) + Lv_2(t).$$

The resulting transfer functions from r and noises v_1 and v_2 to the estimation error e_1 are

$$H_r(s) \equiv 0, \qquad H_1(s) = \frac{1}{s+L}, \qquad H_2(s) = -\frac{L^2}{s+L} + L = \frac{Ls}{s+L}.$$

Since the peak value of $|H_1(j\omega)|$ is L (achieved at $\omega = 0$) and the peak value of $|H_2(j\omega)|$ is $1/L$ (achieved as $\omega \to \infty$), the corresponding estimation quality measure equals $J(L) = L + 1/L$. It is easy to see, by taking derivative with respect to L, that $J(L)$ is minimized at $L = 1$. With respect to the performance criterion chosen above, the best optimal estimator is

$$\hat{p}_1 = \hat{p}_* + y_2, \quad \text{where} \quad \dot{p}_* = -p_* + y_1 - y_2.$$

A Better Noise Model

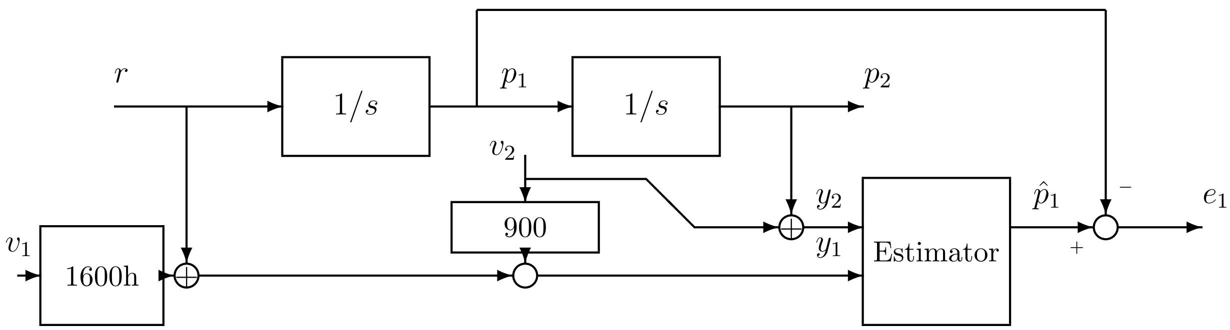
There are some refinements to be done with the example above, mostly concerning the noise part of the model. In the original formulation, using the sum of maximal values of $|H_k(j\omega)|$ corresponds to assuming that the "noises" v_1, v_2 are, in a certain sense, "independent", and have the same "intensity". Neither assumption is reasonable, though:

- Since $r = 1600x_1 + 900x_3$ is not measured directly (only x_1 and x_3 are), one should introduce v_1 as the noise associated with measuring x_1, not r, which leads to an alternative model of noise insertion:

$$\begin{aligned} y_1(t) &= r(t) + 1600v_1(t) + 900v_2(t), \\ y_2(t) &= p_2(t) + v_2(t). \end{aligned}$$

- Since the sensors of x_1 and x_3 are of very different physical nature, the corresponding measurement noise levels should be expected to be different, too. This can be reflected by replacing $v_1(t)$ by $hv_1(t)$ in the sensor equations, where h is the constant reflecting the ratio of current and position sensing noises intensity. Accordingly, the final sensing model will become

$$\begin{aligned} y_1(t) &= r(t) + 1600hv_1(t) + 900v_2(t), \\ y_2(t) &= p_2(t) + v_2(t). \end{aligned}$$



In the new model (with the same estimator as above) the transfer functions H_1 and H_2 (from v_1 and v_2 to e) become

$$H_1(s) = \frac{1600h}{s+L}, \qquad H_2(s) = \frac{Ls+900}{s+L}.$$

The corresponding estimation quality measure equals

$$\frac{1600h}{L} + \max\left\{L,\ \frac{900}{L}\right\},$$

and is minimized at $L = 30$ for all $0 \le h \le \frac{9}{16}$, or at $L = 40\sqrt{h}$ for all $h \ge \frac{9}{16}$.