6.310 4/1/26

C.T. State-Space Continued

Standard Format A, B, C, D matrices

$$\frac{d}{dt}x(t) = A\,x(t) + B\,u(t) \qquad y(t) = C\,x(t)$$

$$\begin{bmatrix}\dot{x}(t)\end{bmatrix} = N\updownarrow\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix}x(t)\end{bmatrix} + \begin{bmatrix}B\end{bmatrix}\begin{bmatrix}u(t)\end{bmatrix} \qquad \underbrace{\begin{bmatrix}y(t)\end{bmatrix} = \begin{bmatrix} C \end{bmatrix}\begin{bmatrix}x(t)\end{bmatrix}}_{y(t) = \sum_{i=1}^{N} c_i x_i(t)}$$

$\longleftrightarrow$ $N = \#$ states

Sometimes there is a D

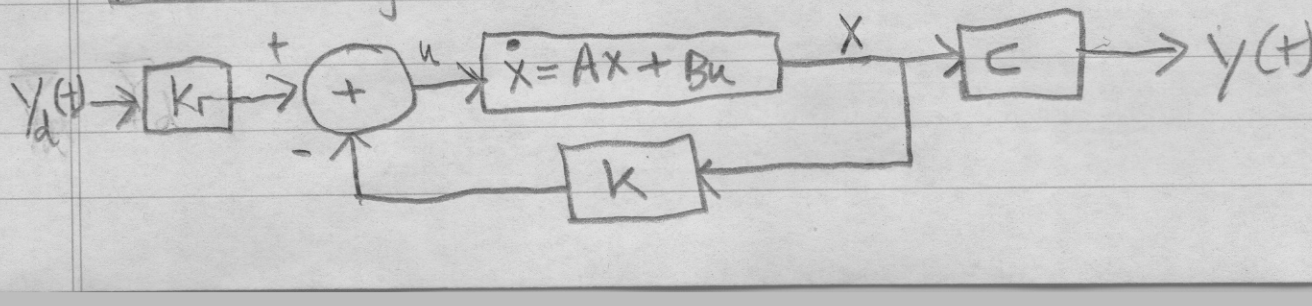
$$y(t) = Cx(t) + Du(t)$$

Feedback

$$u(t) = K_r\,y_d(t) + K\,x(t)$$

$$\begin{bmatrix}u(t)\end{bmatrix} = \begin{bmatrix}K_r\end{bmatrix}\begin{bmatrix}y_d(t)\end{bmatrix} + \underbrace{\begin{bmatrix}K_1, \dots, K_N\end{bmatrix}\begin{bmatrix}x_1(t) \\ \vdots \\ x_N(t)\end{bmatrix}}_{\sum_{i=1}^{N} K_i x_i(t)}$$

Block Diagram



State - Space Zero Input Response

$$\dot{x} = Ax + Bu \qquad u = \underbrace{K_r y_d}_{=0} - Kx$$

$$\dot{x} = (A - BK)x \qquad x(0) \neq 0$$

$$\begin{bmatrix} A \end{bmatrix} - \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} A - BK \end{bmatrix}$$

$$x(t) = e^{(A-BK)t}\, x(0)$$

eigenvalues $(A-BK)$

$$x(t) = \begin{bmatrix} \\ V \\ \\ \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{bmatrix} \begin{bmatrix} \\ V \\ \\ \end{bmatrix}^{-1} \begin{bmatrix} \uparrow \\ x(0) \\ \downarrow \end{bmatrix}$$

matrix of evectors

$= A - BK$ if diagonalizable

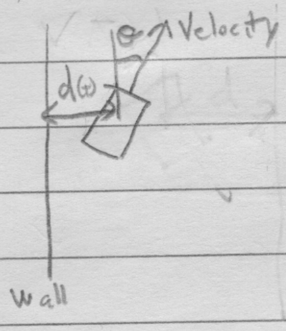
$$\Rightarrow \quad x_i(t) = \sum_{j=1}^{N} \alpha_{i,j}\, e^{\lambda_j t}$$

Want $x_i(t) \to 0$ fast

Pick K's so that real(λ's) $\to -\infty$
so $e^{\lambda_i t} \to 0$ fast for all i.

Example

~~Line following Robot~~



$$\underbrace{\dot{\begin{bmatrix} d \\ \theta \end{bmatrix}}}_{\dot{X}} = \overbrace{\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} d \\ \theta \end{bmatrix}}^{X} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u$$

u = Robot rotational velocity (we control)

$$= -KX = -[K_1 \ K_2] \begin{bmatrix} d \\ \theta \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \ K_2]$$

$$= \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & V \\ -K_1 & -K_2 \end{bmatrix}$$

eigs$(A-BK)$: $\det\left(\underbrace{\begin{bmatrix} \lambda & -V \\ K_1 & \lambda + K_2 \end{bmatrix}}_{(\lambda I - (A - BK))} \right) = 0$

$$\Rightarrow \quad \lambda(\lambda + K_2) + K_1 V = 0$$

$$\lambda^2 + K_2 \lambda + K_1 V = 0$$

Lets pick $\lambda_1 = -100$ & $\lambda_2 = -200$

For $\lambda_1 = -100$ ($e^{-\lambda_1 t}, e^{\lambda_2 t}$ both go to zero in hundredths of a second)
$\lambda_2 = -200$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + (300)\lambda + 20{,}000 = 0$$

$$\lambda^2 + K_2 \lambda + K_1 V = 0$$

$K_2 = 300 \quad K_1 = 20000/V = 20{,}000$ if $V = 1$

So if robot is $\underbrace{0.01_m}_{1\,cm}$ from wall (but parallel, so $\theta = 0$)

then $u = 200$ radians/sec

Mr. Robot's gonna be dizzy.

We can place the λ's anywhere we want by adjusting K, but the result may be a controller whose u's are unrealisticly large!

Only for "good" matrix pairs A, B

State-Space Controller Tradeoff

$Re(eig(A-BK))$ as negative as possible (fast response)

$|-Kx|$ less ~~than~~ actuator MAXIMUM (feasible)

or generally

"Better Dynamics"

"lower energy"

Example Senario

λ's ← ~~natural freqs~~

You pick two poles for the line follower e.g. -2 and -5

But $|Kx|$ > maximum robot rotation rate

Ques

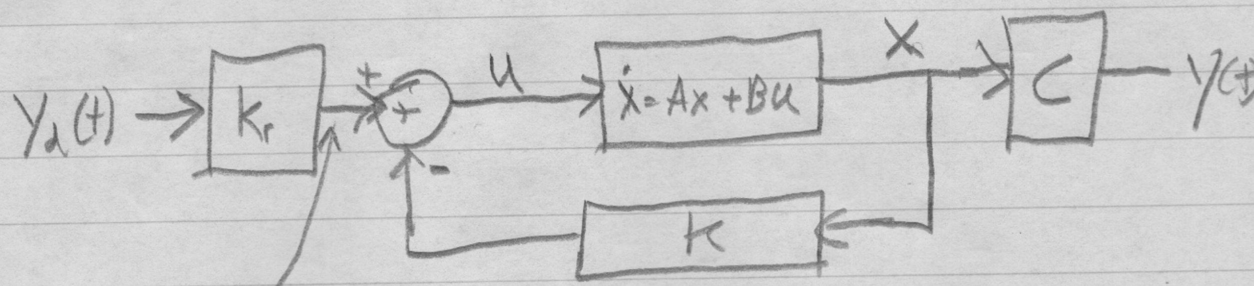
How do you change the poles (λ's) to reduce the command $|Kx|$? Should you change one λ or both?

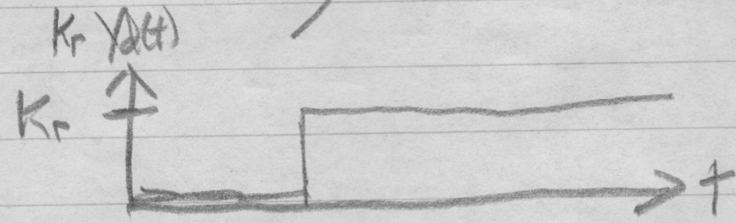
Soln

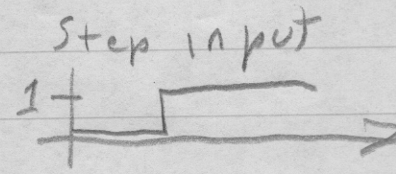
Brute Force → Build a model, use simulation to try many alternatives (search the λ space), noting $|Kx|$ for each candidate!

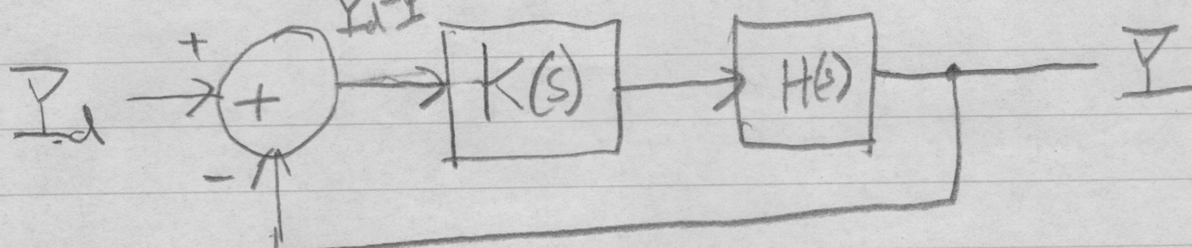
What if Model is wrong?

Aside: Compare S.S. to T.F.

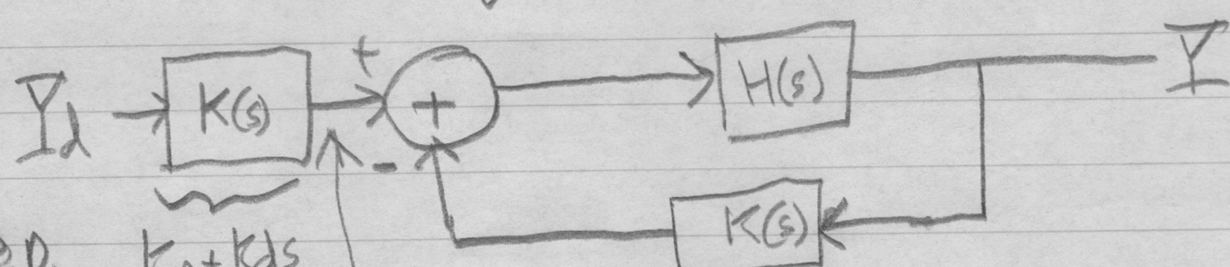




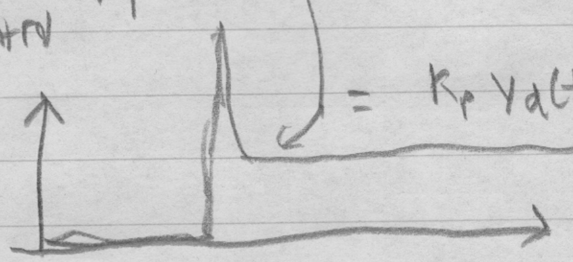




⇓ By Linearity



P.D. Control $\quad K_p + K_d s$



$= K_p y_d(t) + K_d \frac{d}{dt} y_d(t)$