

PostLab 5: Win, Place, LQR – Solutions

In this post lab, we will analyze disturbance rejection in state space systems. Recall the general form of state space system is given by:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where we assume the E matrix is identity. Now suppose there is an unmodelled disturbance that can affect the system state variables $x(t)$. The state space system is modified:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Fd(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Here $d(t)$ is the disturbance, and F is an $n \times 1$ vector that maps a scalar disturbance to the state variable $x(t)$. Equivalently, the new system has two input variables: $u(t)$ and $d(t)$. Suppose we define a new input vector:

$$u'(t) = \begin{bmatrix} u(t) \\ d(t) \end{bmatrix}$$

Based on this new input vector, we want to write an equivalent state space system:

$$\begin{aligned}\dot{x}(t) &= A'x(t) + B'u'(t) \\ y(t) &= C'x(t) + D'u'(t)\end{aligned}$$

that has the same state variables $x(t)$ and output $y(t)$. Please write the new matrices A' , B' , C' , and D' in terms of the original matrices A , B , C , D , and F . In addition, please write the dimensions of each matrix. Here $x(t)$ has dimension of $n \times 1$.

$A' =$

$A : n \times n$ matrix

$B' =$

$[B \ F] : n \times 2$ matrix

$C' =$

$C : 1 \times n$ matrix

$D' =$

$[D \ 0] : 1 \times 2$ matrix

Now let's implement a state space controller for designing the input vector $u'(t)$. We let

$$u'(t) = \begin{bmatrix} K_r r \\ d \end{bmatrix} - K'x$$

Here d is the disturbance that we do not have direct control. What is the dimension of the feedback matrix K' ?

dimension of K' :

$2 \times n$ matrix

Let $x(t)$, $y(t)$, and $d_{dist}(t)$ be represented by their transforms:

$$x(t) \rightarrow X(s); \quad y(t) \rightarrow Y(s); \quad d(t) \rightarrow D_{dist}(s)$$

and solve for the state $X(s)$ of the new controller as a function of the disturbance $D_{dist}(s)$. You can assume $r(t) = 0$. Enter your work in the box below.

$$\dot{x}(t) = A'x(t) + B'u'(t)$$

$$\dot{x}(t) = A'x(t) + B' \left(\begin{bmatrix} K_r r \\ d \end{bmatrix} - K'x \right)$$

$$sX = A'X + B' \left(\begin{bmatrix} K_r R \\ D_{dist} \end{bmatrix} - K'X \right)$$

$$(sI - (A' - B'K')) X = B' \begin{bmatrix} K_r R \\ D_{dist} \end{bmatrix}$$

$$X = (sI - (A' - B'K'))^{-1} B \begin{bmatrix} K_r R \\ D_{dist} \end{bmatrix}$$

$$X = (sI - (A' - B'K'))^{-1} [B \ F] \begin{bmatrix} K_r R \\ D_{dist} \end{bmatrix}$$

$$X = (sI - (A' - B'K'))^{-1} (BK_r R + F D_{dist})$$

$$X = (sI - (A' - B'K'))^{-1} F D_{dist}$$

Note that in the last step we use $R(s) = 0$.

Next, find the closed-loop transfer function between the output $Y(s)$ and the disturbance $D_{dist}(s)$.

$$y = C'x + D'u = C'(sI - (A' - B'K'))^{-1} F D_{dist}$$

$$H_{close} = C'(sI - (A' - B'K'))^{-1} F$$

Now having the equations, we are going to explore three cases computationally. We will use lab 5 as an example. Specifically, please generate the A , B , C , and D matrices in lab 5. For ease of grading, please use the following parameters:

$$\lambda_e = -125; \quad \gamma_{ic} = 1.1; \quad \gamma_{ai} = 900; \quad \gamma_{ay} = 1000$$

We will assume an external disturbance directly causes a change of the displacement state Δy . So the F matrix is given by:

$$F = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In the first case, we are going to use pole placement just as we did in lab 5. We will completely ignore the matrix F . So you should design your K matrix using the command:

`K = place(A, B, poles)`

For ease of grading, please use the pole vector

`poles = [-190, -200, -20]`

Please use this K vector to simulate the system response to a step disturbance. What are the A , B , C , D , and K matrices?

$$A = \begin{bmatrix} -125 & 0 & 0 \\ 900 & 0 & 1000 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 137.5 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0 \ 0 \ 1]$$

$$D = 0$$

$$K = [2.0727 \ 0.3782 \ 9.4545]$$

In the second case, let's design a new K matrix using knowledge of F . Specifically, use the A' and B' matrices you found above to design a new K matrix. Please write the new K matrix.

Hint: the size of this K matrix should be 2×3 .

$$K = \begin{bmatrix} 0.5913 & 0.0170 & 0.1312 \\ 12.3693 & 2.8995 & 203.7013 \end{bmatrix}$$

In the third case, please use the integral form of the state space controller with LQR. When formulating the control matrix, please use the following Q and R matrices:

$$Q = \begin{bmatrix} 2000 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = 1$$

What are the A_+ , B_+ , C_+ , D_+ , and K_+ matrices?

$$A_+ = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -125 & 0 & 0 \\ 0 & 900 & 0 & 1000 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_+ = \begin{bmatrix} 0 \\ 137.5 \\ 0 \\ 0 \end{bmatrix}$$

$$C_+ = [0 \ 0 \ 0 \ 1]$$

$$D_+ = 0$$

$$K_+ = [44.7214 \ 1.1803 \ 0.1940 \ 6.7805]$$

Please simulate the system response to a step disturbance for all three cases. Please superimpose all three plots on each other. You should see the original controller has the worst disturbance rejection, the controller that considers the F vector performs significantly better, and the integral LQR controller can remove the steady state error.

